A NEW ALGORITHM FOR FREQUENCY DOMAIN ANALYSIS OF NONLINEAR CIRCUITS

FLOREA IOAN HĂNŢILĂ, FLORIN CONSTANTINESCU, ALEXANDRU GABRIEL GHEORGHE, MIRUNA NIŢESCU, MIHAI MARICARU

Key words: Steady state, Nonlinear circuit analysis.

Consider a dynamic circuit containing nonlinear resistors, the other elements being linear. Each nonlinear resistor is replaced by a set of equivalent sources which are current or voltage dependent. The parameters of these sources are corrected iteratively. In order to find the new controlling voltages (currents) linear sinusoidal problems are solved for source harmonics. The convergence of this iterative procedure is proved. This method gives correct results for strongly nonlinear circuits driven by modulated signals, unlike the harmonic balance method. An example is given for illustration.

1. INTRODUCTION

The periodic steady state computation of nonlinear circuits can be often very time consuming and/or can exhibit convergence difficulties. The time domain analysis using brute force cannot be used for circuits with modulated signals due to the huge number of points needed to be swept till transient components decay. A partial solution to this difficulty is shooting which speeds up the finding of the steady state using several Newton-Raphson iterations [1]. Shooting algorithms are implemented in circuit simulators SPECTRE RF [1] and PAN [2]. The two time variable analysis, aimed to jump over some carrier periods, can lead to better computation efficiency if shooting on both fast and slow time scales is used [3]; unfortunately the computation of the jump on the slow time scale remains an open problem. Despite its efficiency, harmonic balance, the most known frequency domain analysis method, gives correct results only for weak or mild nonlinear circuits. Harmonic balance is implemented in ADS, APLAC, and SPECTRE RF circuit simulators.

A new algorithm for frequency domain analysis of circuits with resistive nonlinearities is presented in this paper. The nonlinearity is treated iteratively by a
fixed point method [4], successfully used for solving nonlinear electromagnetic field problems. The nonlinear resistors are replaced by equivalent voltage (current) dependent sources. By this way the nonlinearities are “transferred” to the sources, the circuit impedances being linear. Using a Fourier decomposition of the sources, each harmonic of the controlling voltages (currents) is computed by solving a linear circuit. These harmonics are used to correct the sources a. s. o. It can be proved that this iterative numerical process is convergent. The number of linear systems of equations that has to be solved at each iteration step is given by the number of harmonics taken into account. The iterations can be started with a small number of harmonics in order to increase the computation efficiency.

2. EQUIVALENT SOURCES

We consider that the p-port resistor constitutive relationship \( i = F(u) \), where \( F : R^p \to R^p \) is Lipschitzian

\[
\| F(u') - F(u'') \|_{R^p} \leq \Lambda \| u' - u'' \|_{R^p}, \quad \forall u', u'' \in R^p
\]

and monotonic

\[
\langle F(u') - F(u''), u' - u'' \rangle_{R^p} \geq \lambda \| u' - u'' \|^2_{R^p}, \quad \forall u', u'' \in R^p,
\]

with \( \lambda > 0 \), where the inner product is \( \langle x, y \rangle_{R^p} = x^T y \).

![Fig. 1. Two-port nonlinear resistor.](image)

![Fig. 2. Equivalent sources for a two-port nonlinear resistor.](image)

The resistor may be replaced at any port, by the sources

\[
u_k = r_k i_k + e_k,
\]

where \( e_k \) has a nonlinear dependence on \( u_k \) [4],

\[
e_k = u_k - r_k F_k(u) \equiv G_k(u).
\]
If we choose $r_k = \alpha$ with

$$\alpha \in \left(0, \frac{2\lambda}{\Lambda^2}\right),$$  \hspace{1cm} (5)

then the function $G$ is a contraction, i.e.

$$\|G(u') - G(u'')\|_{H^p_{x,y}} \leq \theta \|u' - u''\|_{H^p_{x,y}},$$  \hspace{1cm} (6)

where the contraction factor is

$$\theta = \sqrt{1 - 2\alpha\lambda + \alpha^2 \Lambda^2} < 1$$  \hspace{1cm} (7)

(see Appendix 1). The smallest value $\theta = \sqrt{1 - \lambda^2 / \Lambda^2}$ of the contraction factor is obtained for $\alpha = \lambda / \Lambda^2$. In the case of the one-port nonlinear resistor, equations (1) and (2) lead to a positive bounded increasing $u$-$i$ relationship and

$$\theta = \text{Max}(1 - \alpha \cdot g_{\min}, \alpha \cdot g_{\max} - 1),$$

with $g_{\min} = \inf_{u'} \left[ (F(u') - F(u''))/(u' - u'') \right]$, $g_{\max} = \sup_{u''} \left[ (F(u') - F(u''))/(u' - u'') \right]$.

The circuit may contain many nonlinear resistors whose equivalent circuits are similarly built. In the following we consider, for the sake of simplicity, that the circuit contains only one p-port nonlinear resistor. The following results, obtained using this assumption, remain valid for the case of more than one p-port nonlinear resistor.

3. PERIODIC SOLUTION OF THE LINEAR CIRCUIT

Replacing the nonlinear resistors with voltage (or current) dependent sources, we obtain a linear circuit with “linear” relationship (3) for resistors. Let $u'_\beta$, $u''_\beta$, and $i'_\beta$, $i''_\beta$ be the branch voltages and currents, respectively, due to the sources $e'$, $e''$ and $\Delta u_\beta = u'_\beta - u''_\beta$, $\Delta i_\beta = i'_\beta - i''_\beta$, $\Delta e_\beta = e'_\beta - e''_\beta$. Tellegen’s theorem gives

$$\begin{align*}
\langle \Delta u_\beta, \Delta i_\beta \rangle_x &= \langle \Delta u_\beta, g \Delta u_\beta - g \Delta e \rangle_y + \\
+ \langle \Delta u_t, C \frac{d \Delta u_t}{dt} \rangle_y + \langle L \frac{d \Delta i_t}{dt}, \Delta i_t \rangle_y = 0,
\end{align*}$$

where $u_\beta$, $u_t$ are the vectors of the resistor and capacitor voltages and $i_\beta$ is the vector of the inductor currents; $r, c, l, b$ are the numbers of the resistors, capacitors,
inductors and of all branches respectively; $C$ and $L$ are positively defined matrices of the capacitances and inductances respectively, and $g = r^{-1}$ is the symmetric and positively defined matrix of all conductances. The vector $\Delta e$ has nonzero entries only for nonlinear resistor ports. Therefore, for the periodic solution we have

$$\int_0^T \langle \Delta u_\nu, g \Delta u_\nu \rangle_{K'} \ dt \leq \int_0^T \langle \Delta u_\nu, g \Delta e \rangle_{K'} \ dt$$

and

$$\int_0^T \langle \Delta u_\nu, g \Delta e \rangle_{K'} \ dt \leq \int_0^T \langle \Delta e, g \Delta e \rangle_{K'} \ dt .$$

It follows that the function $e \xrightarrow{w} u$, giving the periodic solution of the linear circuit, is non-expansive in the Hilbert space of the periodic functions.

4. FOURIER ANALYSIS

Any periodic sources $e_k$ has a Fourier series expansion in the form

$$e_k(t) = \sum_n \left( (e'_k)_n \sin(n\omega t) + (e''_k)_n \cos(n\omega t) \right).$$

For the numerical computation, we retain only a finite number $N$ of harmonics, namely $e \simeq e_a = Y(e)$, the approximation $Y$ being non-expansive. For each harmonic of $e_a$ the corresponding complex voltage is considered.

5. ITERATIVE PROCEDURE

Starting from arbitrary initial values for the sources in (3) the port voltages of the nonlinear resistor are computed. If the null initial value is chosen, we obtain:

$$u(t) = u^{(0)} = W(0).$$

These voltages are produced by the independent sources in the circuit. It follows the sources correction using the relation (4):

$$e(t) = e^{(1)} = G(u^{(0)}).$$
The harmonic components of these sources are computed and some of them are used for further computation. The harmonic components of the resistor port voltages are computed as:

\[
\begin{pmatrix}
(u'_k)_n \\
(u''_k)_n
\end{pmatrix} = \begin{pmatrix}
(a_{k_{11}})_n & (a_{k_{12}})_n & (a_{k_{21}})_n & (a_{k_{22}})_n \\
(e'_k)_n & (e''_k)_n
\end{pmatrix} \cdot \begin{pmatrix}
(nk) \\
(nk)
\end{pmatrix}.
\] (14)

These harmonics are produced by the resistor dependent sources.

Using the superposition theorem in the linear equivalent circuit, the time domain port voltages \( u^{(i)}_k \) of the nonlinear resistor are computed. These values are obtained as a sum of the voltages \( u^{(0)} \) produced by the independent sources and the time domain waveforms corresponding to the result in (12).

The above steps are repeated until the difference between the last two values of the resistor port voltages is small enough. Each step may be outlined as:

\[
e^{(m)} \xrightarrow{Y} e^{(m)}_a \xrightarrow{W} u^{(m)} \xrightarrow{G} e^{(m+1)}.
\] (15)

Because the functions \( W \) and \( Y \) are non-expansive, and \( G \) is a contraction, the composition of these three functions is a contraction having the contraction factor \( \theta \). It follows that the above iterative procedure generates a Picard-Banach sequence which converges to the fixed point of the composed mapping.

**Remarks:**

1) The initial value \( u^{(0)} = W(0) \) and the matrices \( (a_k)_n \) in (14) are computed only once, before starting the iterations.

2) In most circuits the number of the nonlinear resistor ports is smaller than the resistor branches number, because of the linear resistors. Using (10) it follows:

\[
\int_0^r \langle \Delta u, g \Delta u \rangle_{R^r} dt < \int_0^r \langle \Delta u_p, g \Delta u_n \rangle_{R^r} dt \leq \int_0^r \langle \Delta e, g \Delta e \rangle_{R^r} dt.
\] (16)

The inequality defining the non-expansive character of the mapping \( L \) is stronger in this case, and the convergence speed of the iterative procedure is increased.

3) As the harmonics number taken into account in the Fourier series is smaller, the inequality \( \int_0^r \langle y(e_k) \rangle dt \leq \int_0^r e_k^2 dt \), defining the non-expansive character of the mapping \( Y \) is stronger. This fact leads to an increased speed of the iterative procedure, too.
6. HARMONIC COMPONENT SELECTION

The 3-rd remark suggests a method for the convergence speed improvement through the selection of the most outstanding weight harmonics. In the first iteration all harmonic components of \( e^{(1)} \) up to a certain range \( N \) are computed. Starting with the second iteration only those harmonic components having a weight greater than a prescribed value are selected:

\[
\frac{\sqrt{(e_k^2)^{n} + (\epsilon_k^2)^{n}}}{E} \geq q_1, \tag{17}
\]

where \( E \) is a reference value, for example

\[
E = \sqrt{\frac{2}{T} \int_{0}^{T} \| u^{(n)} \|_{e}^{2} \, dt}. \tag{18}
\]

The iterative procedure is swept using the selected harmonic components only. The Fourier analysis is much faster than in the case in which all harmonic components are taken into account. Also the contractive character of the function chain \( YWG \) is stronger and the convergence speed is increased. The iterations are stopped if the error is smaller than a prescribed value

\[
\text{err} = \frac{1}{E} \sqrt{\frac{1}{T} \int_{0}^{T} \| \Delta u^{(n+1)} \|_{e}^{2} \, dt} \leq e_1. \tag{19}
\]

The Fourier analysis with \( N \) harmonic components can be made again, selecting only those components having a weight \( q_2 < q_1 \). Using these harmonics the iterations are stopped if the error \( e_1 \) is reached. Considering more harmonics and having a weaker contractive character of the function chain \( YWG \), the convergence speed decreases. The harmonics selection algorithm can be stopped if the iteration number for the new selection is small (for example smaller than eight iterations), therefore the improvement obtained by increasing the harmonics number is not significant.

7. AM DEMODULATOR

Consider the circuit in Fig. 3 with \( R = 1 \, \text{k}\Omega, \ C = 1 \, \mu\text{F}, \ e(t) = 2 \sin(2\pi f_1) \sin(2\pi f_2), \ f_1 = 1 \, \text{kHz}, \ f_2 = 1 \, \text{MHz} \) and the diode \( u-i \) relationship given in Fig. 4.
The nonlinear element is replaced by a 10 Ω resistor in series with a voltage source, whose dependence on the diode branch voltage $u$ is

$$e = G(u) = \begin{cases} 0, & \text{for } u \geq 0 \\ 0.99999u, & \text{for } u < 0. \end{cases} \quad (20)$$

The value $E = 2$ is considered in the relations (17) and (19). The data describing a set of harmonics selection are given in Table 1.

<table>
<thead>
<tr>
<th>Selection</th>
<th>Weight</th>
<th>Harmonics number</th>
<th>Iteration number</th>
<th>Err</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.01</td>
<td>16</td>
<td>78</td>
<td>$0.890 \cdot 10^{-7}$</td>
</tr>
<tr>
<td>2</td>
<td>0.005</td>
<td>19</td>
<td>27</td>
<td>$0.841 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>3</td>
<td>0.002</td>
<td>32</td>
<td>11</td>
<td>$0.882 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>4</td>
<td>0.001</td>
<td>41</td>
<td>31</td>
<td>$0.945 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>5</td>
<td>0.0005</td>
<td>85</td>
<td>55</td>
<td>$0.984 \cdot 10^{-7}$</td>
</tr>
</tbody>
</table>

A part of the diode voltage plot is given in Fig. 5.
The capacitor voltage is shown in Fig. 6 and Fig. Also 7. Fig. 7 shows a high frequency detail.

![Fig. 6 – Capacitor voltage. Fig. 7 – Capacitor voltage (detail).](image)

If 16, 41, 85 or 10000 harmonic components are used the plots in Fig. 5 and Fig. 7 remain the same. A detail of Fig. 7, given in Fig. 8, illustrates the small differences between various harmonics selections. A 0.7‰ error can be observed between the results obtained with 85 and 10000 harmonics.

The result obtained with the proposed algorithm is compared with the result of harmonic balance and shooting with Newton-Raphson. The capacitor voltage waveforms given by these methods are shown in Fig. 9.

![Fig. 8 – Comparison between various harmonics selections.](image)

![Fig. 9 – $u_c$ given by the proposed method, APLAC, and SPECTRE RF.](image)
The harmonic balance response is obtained with APLAC, while the shooting response is computed with the PSS analysis of SPECTRE RF. A very good agreement between the proposed algorithm and the shooting method can be observed. The differences between these two waveforms and the harmonic balance response can be explained by the convergence problems of this last method in the case of strong nonlinearities.

8. CONCLUSIONS

A new algorithm for frequency domain analysis of circuits with nonlinear resistors has been presented in this paper. The convergence of the proposed iterative procedure has been proved. An automated selection of the harmonic components whose weight exceeds an imposed limit can be performed in order to improve the convergence speed and the computation efficiency. This algorithm converges for circuits with strong nonlinearities, unlike the harmonic balance which has convergence problems in this case.

Further research will include efficient FFT algorithms implementation and test of more intricate circuits.

APPENDIX 1

Taking (3) into account we have:

\[ \theta^2 = \sup_{s' < s} \frac{\|G(u') - G(u'')\|_w}{\|u' - u''\|_w} = \]
\[ 1 - 2\alpha \left( \frac{\langle u' - u'', F(u') - F(u'') \rangle_{R^p}}{\|u' - u''\|_{R^p}^2} \right)^\alpha + \alpha^2 \frac{\|F(u') - F(u'')\|_{R^p}^2}{\|u' - u''\|_{R^p}^2} \]

and, according to (1) and (2), it follows (7). For the one–port resistor the relationships (2) and (1) show that \( F \) is uniformly and bounded increasing:

\[ \frac{F(u') - F(u'')}{u' - u''} > \lambda, \quad \frac{F(u') - F(u'')}{u' - u''} < \Lambda, \]

(22)

It follows:

\[ \theta = \sup_{s' < s} \left| \frac{G(u') - G(u'')}{u' - u''} \right| = \sup_{s' < s'} \left| -\frac{\alpha F(u') - F(u'')}{u' - u''} \right| = \text{Max}(1 - \alpha \cdot g_{\text{max}}, \alpha \cdot g_{\text{max}} - 1). \]

Received on July 14, 2008
REFERENCES


2. ***PAN simulator, to be downloaded free from http://brambilla.elet.polimi.it.***
