COMPUTATION OF THE 3D TEMPERATURE FIELD INSIDE THE MOVING BEAMS USING EIGENFUNCTIONS SERIES

MIHAI MARICARU[1], GEORGE-MARIAN VASILESCU [1], TEODOR LEUCA[2], IOAN FLOREA HĂNȚILĂ[1], BOGDAN-CIPRIAN STANCIU[3]

Key words: Analytical method, Space eigenfunctions expansion, Eddy current continuous flow heating.

The cross-section of prefabricated metallic beams is constant along the path they move. This feature allows solving analytically the temperature field problem by applying the method of spatial eigenfunctions series decomposition in the cross-section. The second order differential equations of the series' terms can be solved analytically. A Picard-Banach iterative technique with a very fast convergence rate is proposed for solving the eigenvalues problem. The method has a greater efficiency in terms of accuracy and execution times than numerical methods, such that based on the finite element method. It can be shown that when the speed of moving beam is sufficiently small the distribution of the field in the cross-section is almost uniform. For this case a simple and fast procedure which allows the computation of the average temperature variation along the beam is proposed.

1. INTRODUCTION

Metallic materials can be processed with more ease when they are heated. For an increased productivity the heating of the pre-fabricated beams is done with electromagnetic fields in continuous flow (Fig. 1). The beam enters with velocity $\boldsymbol{v}$, in a sinusoidal current excited coil that induces eddy currents in the beam which, in turn, heat the beam (Fig.1). The mathematical model of the beam heating involves the solution of an eddy current problem and a temperature field problem. In a previous paper [1], which can be considered the first part of this study, an efficient analytical method for solving the electromagnetic field is presented. The method is based on the expansion of the solution in a series of spatial eigenfunctions. It was proposed by Mihai Vasiliu [2–4] and, in comparison with numerical methods, has

[1] University "Politehnica" of Bucharest, 313 Splaiul Independenței, 060042 Bucharest, Romania, mihai.maricaru@upb.ro

[2] University of Oradea, Faculty of Electrical Engineering and Information Technology, 410087 Oradea, Romania

[3] SC UAMT Oradea, Romania

Rev. Roum. Sci. Techn. – Électrotechn. et Énerg., **59**, *2,* p.119–129, Bucharest, 2014

the great advantage of a more accurate solution when the electromagnetic field depth of penetration is small. The formulas for calculating the specific losses and global cross-section losses have been established in [1].

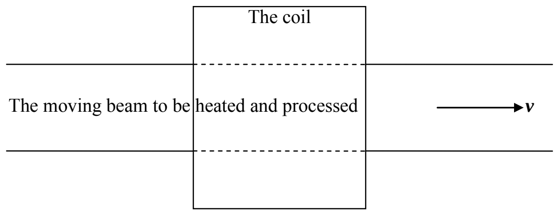


Fig. 1 – Continuous flow heating device.

The Mihai Vasiliu method is also used in this paper to solve the 3D moving beams heating problem. The solution of the temperature field is expanded in a series of spatial eigenfunctions of the cross section. The series' coefficients are functions of the z coordinate defined on the beam's axis. These coefficients are obtained by analytically solving a set of differential equations of the second degree. Unlike [1], computation of the eigenvalues leads to finding solutions to a set of transcendental equations. In this paper a method is proposed in which these equations are changed into contractive operators whose fixed points are rapidly determined through the Picard-Banach iterative procedure.

The small depth of penetration leads to a strongly non-uniform distribution of the specific losses. For this reason the discretization mesh used in the numerical methods (such as the finite element method in 3D) is strongly non-uniform with unwanted consequences for the size and conditioning of the system's matrix. The proposed method has a greater accuracy and a smaller execution time than those of numerical methods and is not influenced by the non-uniform distribution of the specific losses. The method can be easily applied for the computation of the 3D temperature field of moving beams.

A one-dimensional model is also proposed for the case in which the temperature field is almost uniform in the cross-section, the temperature of the beam changes only in the displacement direction. The great advantage of this model is that presents small execution times, it does not require obtaining the eigenvalues and eigenfunctions, and it can be applied to any shape of the cross section. The model is recommended for the design of an optimal constructive solution for various beams.

2. THE EQUATION OF THE TEMPERATURE FIELD

In the coil's frame of reference the equation of the temperature field is

$$-\nabla\cdot\lambda\nabla\theta+c\frac{\mathrm{d}_s\theta}{\mathrm{d}t}=p\,, \tag{1}$$

where θ is the temperature of the beam (we assume the exterior temperature is equal to zero); λ is the thermal conductivity; *c* is the volumetric heat capacity; *p* are the specific losses (produced by the eddy currents); $\frac{\mathrm{d}_s\theta}{\mathrm{d}t} = \frac{\partial\theta}{\partial t} + \mathbf{V}\cdot\nabla\theta = \frac{\partial\theta}{\partial t} + V\frac{\partial\theta}{\partial z}$ is the substantial derivative of the temperature. Due to the fact that the device performs the heating in continuous flow, the temperature is constant in the frame of reference of the device and $\frac{\partial\theta}{\partial t} = 0$. For homogenous media ($\lambda$ = const., c = const.), equation (1) can be written in the form

$$-\Delta\theta + q\frac{\partial\theta}{\partial z} = \frac{p}{\lambda}, \tag{2}$$

where $q = \frac{cV}{\lambda}$.

On the contour of each cross-section we have the mixed boundary condition

$$\lambda\frac{\partial\theta}{\partial n} + \tau\theta = 0, \tag{3}$$

where τ is the heat transfer coefficient. The temperature at the entry of the beam in the excitation coil is the exterior one

$$\theta = 0, \text{ for } z=0. \tag{4}$$

When exiting the coil the beam's temperature is at its maximum

$$\frac{\partial\theta}{\partial z}=0, \text{ for } z = l. \tag{5}$$

3. EXPANDING THE SOLUTION OF THE TEMPERATURE FIELD PROBLEM IN A SERIES OF SPATIAL EIGENFUNCTIONS

In the cross-section Ω_{xoy} the Δ operator's component is $-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. With the boundary conditions from (3) this operator is positive and symmetric. It defines a set of eigenvalues η_k^2 and orthonormal eigenfunctions Ψ_k which verify the equation

$$-\Delta\Psi_k = \eta_k^2\Psi_k. \tag{6}$$

It is worth mentioning that the eigenfunctions Ψ_k are different from the Φ_k functions from the eddy currents problems [1] because the boundary conditions are different. The temperature of the beam is written as

$$\theta(x,y,z)=\sum_k Z_k(z)\Psi_k(x,y)\,. \tag{7}$$

By replacing (7) in equation (2) we have

$$\sum_k\left(Z_k(z)\eta_k^2\Psi_k-\frac{\mathrm{d}^2Z_k}{\mathrm{d}z^2}\Psi_k+q\frac{\mathrm{d}Z_k}{\mathrm{d}z}\Psi_k\right)=\frac{p}{\lambda}\,. \tag{8}$$

By multiplying the relation with Ψ_k and integrating on Ω_{xoy}, we obtain the differential equation of the Z_k component

$$\frac{\mathrm{d}^2Z_k}{\mathrm{d}z^2}-q\frac{\mathrm{d}Z_k}{\mathrm{d}z}-\eta_k^2 Z_k=-p_k\,, \tag{9}$$

where $p_k=\dfrac{1}{\lambda}\int\limits_{\Omega_{xoy}}p\Psi_k\mathrm{d}S$ is obtained through numerical integration, by adopting a discretization network adapted to the distribution of the losses. The solution of the equation (9), that satisfies the boundary conditions (4) and (5), is:

$$Z_k=\frac{p_k}{\eta_k^2}\left[1-\mathrm{e}^{-\frac{\eta_k^2 z}{q_k''+q'}}\left(1+\frac{\eta_k^2}{(q_k''+q')^2}\mathrm{e}^{-2q_k''(l-z)}\right)\Bigg/\left(1+\frac{\eta_k^2}{(q_k''+q')^2}\mathrm{e}^{-2q_k''l}\right)\right], \tag{10}$$

where $q'=q/2$ and $q_k''=\sqrt{q'^2+\eta_k^2}$. The solution of the temperature field problem is obtained by replacing (10) in (7).

4. **NUMERICAL EXAMPLE**

We consider an aluminum beam $2a\times 2b=20\times 80$ mm (Fig. 2), having $\sigma=37{,}7\cdot 10^6$ S/m and $\mu_r=1$. The excitation coil has N windings, a length l = 800 mm, and a current of rms value I and frequency f = 10 kHz [1].

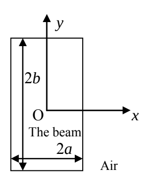


Fig. 2 – The rectangular beam.

A depth of penetration of 0.81969 mm is thus obtained. In the sinusoidal regime the specific losses are given [1] by

$$p=\left(I\frac{N}{l}\right)^2\left[\sum_k\tilde{\Phi}_k\nabla\Phi_k\Big/\left(1-\mathrm{j}\frac{\lambda_k^2}{\omega\mu\sigma}\right)\right]\left[\sum_k\tilde{\Phi}_k\nabla\Phi_k\Big/\left(1+\mathrm{j}\frac{\lambda_k^2}{\omega\mu\sigma}\right)\right], \tag{11}$$

where λ_k^2 are the eigenvalues and Φ_k are the eigenfunctions with null values on the boundary, verifying the equation $-\Delta\Phi_k=\lambda_k^2\Phi_k$ and $\tilde{\Phi}_k=\int_\Omega\Phi_k\mathrm{d}S$.

For the rectangular cross-section we have [1]: $\lambda_k^2=\lambda_{(m,n)}^2=\xi_m^2+\zeta_n^2$,

$$\Phi_{(m,n)}=\frac{2}{\sqrt{ab}}\cos(\xi_m x)\cos(\zeta_n y), \qquad \tilde{\Phi}_{(m,n)}=\frac{2}{\sqrt{ab}}\frac{(-1)^m}{\xi_m}\frac{(-1)^n}{\zeta_n},$$

where: $\xi_m=(m+1/2)\dfrac{\pi}{b}$ and $\zeta_n=(m+1/2)\dfrac{\pi}{b}$.

4.1. THE EIGENVALUES AND THE EIGENFUNCTIONS IN THE TEMPERATURE PROBLEM

To solve the equation of eigenvalues and eigenvectors (6) we write

$$\text{p=}\Psi_k(x,y)=X(x)Y(y) \tag{12}$$

and by dividing by $X(x)Y(y)$ we obtain $-\dfrac{1}{X}\dfrac{\mathrm{d}^2X}{\mathrm{d}x^2}-\dfrac{1}{Y}\dfrac{\mathrm{d}^2Y}{\mathrm{d}y^2}=\eta_k^2$.

As such

$$-\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2}=\alpha^2 \text{ and } -\frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}=\beta^2 \tag{13}$$

and $\eta_k^2=\alpha^2+\beta^2$. Because the function Ψ_k is even in relation with the *x* and *y* variables we limit ourselves to the first quadrant of the cross-section. The equations (13) have the solutions

$$X(x)=A\cos(\alpha x) \text{ and } Y(y)=B\cos(\beta y). \tag{14}$$

The boundary condition (3) is fulfilled if $\lambda\dfrac{\mathrm{d}X}{\mathrm{d}x}+\tau X=0$ and $\lambda\dfrac{\mathrm{d}Y}{\mathrm{d}y}+\tau Y=0$ from which we get

$$\mathrm{ctg}(\alpha a)=\frac{\alpha\lambda}{\tau} \text{ and } \mathrm{ctg}(\beta b)=\frac{\beta\lambda}{\tau}, \tag{15}$$

with the solutions α_m and β_n. Therefore

$$\eta_k^2 = \eta_{(m,n)}^2 = \alpha_m^2 + \beta_n^2, \tag{16}$$

$$\Psi_k = \Psi_{(m,n)} = D_{(m,n)} \cos(\alpha_m x)\cos(\beta_n y), \tag{17}$$

where $D_{(m,n)} = \dfrac{2}{\sqrt{\left[a + \dfrac{\chi}{\alpha_m^2 + \chi^2}\right]\left[b + \dfrac{\chi}{\beta_n^2 + \chi^2}\right]}}$, with $\chi = \dfrac{\tau}{\lambda}$, for $\int_{\Omega_{xoy}} \Psi_{(m,n)}^2 \mathrm{d}s = 1$.

4.2. SOLVING EQUATIONS (15)

For a rapid solution of equations $\mathrm{ctg}(\alpha a) = \dfrac{\alpha\lambda}{\tau}$ (15), a fixed point technique based on the Picard-Banach [5] sequence is used. Equation (15) has a solution $\alpha_k a$ of the argument for each interval $(k\pi, k\pi + \pi/2)$, $k \in N$. We denote $z_k = \alpha_k a - k\pi$, so

$$\alpha_k = (k\pi + z_k)/a. \tag{18}$$

It follows that the equation $\mathrm{ctg}(z_k) = (k\pi + z_k)/(a\chi)$ has a solution $z_k \in (0, \pi/2)$ for each $k \in N$. We replace this equation with

$$f_k(z_k) = z_k - \mathrm{arcctg}\left[(k\pi + z_k)/(a\chi)\right] = 0. \tag{19}$$

The function f_k is Lipchitzian and monotonically increasing. Indeed, because $z_k \in (0, \pi/2)$, we have $\dfrac{\mathrm{d}f_k}{\mathrm{d}z} \in \left(r_k^{'} + 1, r_k^{''} + 1\right)$, where $r_k^{'} = a\chi / \left[(a\chi)^2 + (k\pi + \pi/2)^2\right]$ and $r_k^{''} = a\chi / \left[(a\chi)^2 + (k\pi)^2\right]$. We can choose $\gamma_k < 2\left[(a\chi)^2 + (k\pi)^2\right]/(a\chi)$ so that the function $g_k(z) = z - \gamma_k f_k(z)$ is a contraction $\left|g_k(z') - g_k(z'')\right| < \delta_k \left|z' - z''\right|$, $\forall z', z'' \in R$ with $\delta_k < 1$. The optimal value for γ_k, is $\gamma_{k,opt} = \dfrac{2}{2 + r_k^{'} + r_k^{''}}$, thus the optimal minimal contraction factor is $\delta_{k,opt} = \dfrac{r_k^{''} - r_k^{'}}{2 + r_k^{''} + r_k^{'}}$. For each integer *k*, the function $g_k(z)$ has a fixed point z_k, obtained with the Picard-Banach sequence: $z_k^{(n+1)} = g(z_k^{(n)}) = z_k^{(n)} - \eta f(z_k^{(n)})$ and from (18) we obtain the eigenvalues α_k.

The convergence can be significantly increased if we use overrelaxation in the Picard-Banach sequence

$$z_k^{(n+1)} = z_k^{(n)} - s\eta f(z_k^{(n)}), \tag{20}$$

with $s > 1$. If we stop the Picard-Banach iterations at iteration n, the error when compared with the exact solution can be evaluated with

$$\left| z_k^{(n)} - z_k \right| < \frac{\delta_k}{1-\delta_k} \left| z_k^{(n)} - z_k^{(n-1)} \right| . \tag{21}$$

The contraction factor value decreases with the order of the eigenvalue. For $k > 3$ at most two iterations are sufficient to obtain an almost zero error ($<10^{-20}$). For large values of the argument (at $k > 3$), we can approximate $\operatorname{arcctg}(z) = 1/z$ and an approximate solution of equation (19) is $z_k = \dfrac{d_k}{1+\sqrt{1+2d_k(k\pi)}}$, for $k > 0$, where $d_k = 2a\chi/(k\pi)$, with which the Picard-Banach sequence can be initialized. For $k = 0$ the Picard-Banach sequence can be initialized with the value $z_0 = \sqrt{a\chi}$.

Table 1

The eigenvalues on the *Ox* axis

k	Eigenvalue	Number of iterations	Error	Overrelaxation
0	1.5	6	2.06E-06	2000.0
1	314.2	1	8.55E-09	1.9
2	628.3	1	1.06E-09	1.9
3	942.5	1	3.20E-10	1.9

In Table 1, in the case of the *Ox* axis, the eigenvalues are presented along with the number of iterations required obtaining the error in relation with the exact solution given by equation (16) and the overrelaxation factors.

4.3. THE TEMPERATURE FIELD

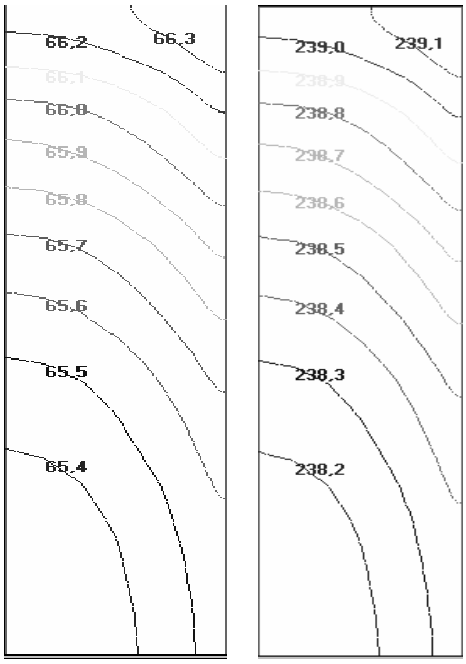
The thermal parameters are $\lambda = 237$, $\tau = 5\ \mathrm{W/^oC/m^2}$, $c = 243 \cdot 10^4\ \mathrm{J/^oC/m^3}$. The excitation coil has an ampere-turn $N \cdot I = 80 \cdot 200 = 16\ \mathrm{kA}$ and a speed of $v = 2$ mm/s. The temperature field is determined with (7). Series (7) has a very fast convergence rate. Only a few terms are required in order to obtain a very good accuracy. This is shown in Table 2 where the *er* errors are indicated at coordinate $z = 0.4$ m computed with the formula $\mathrm{er} = \left| \dfrac{p_{(m,n)}/\eta_{(m,n)}^2}{p_{(0,0)}/\eta_{(0,0)}^2} \right|$. On the horizontal the *m* index is modified and on the vertical the *n* index is modified. With the exception of the $m = 1$ and $n = 1$ indices the errors are almost the same at other *z* coordinates.

Table 2

The error er at $z = 0.4$ m

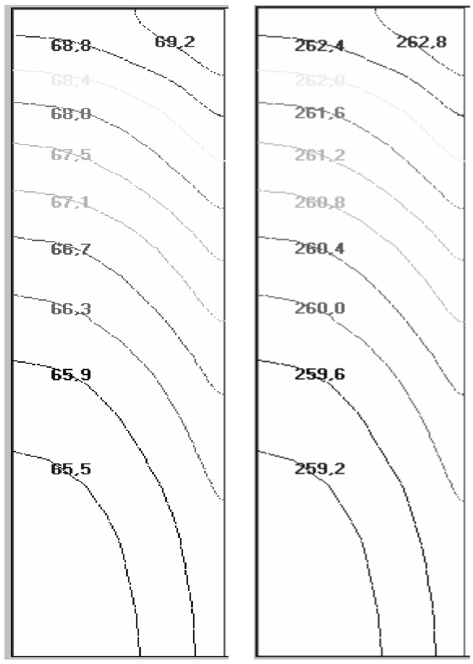
$n\backslash m$	1	2	3	4	5	6	7	8
1	5.0E-02	4.2E-05	9.7E-06	3.9E-06	2.0E-06	1.2E-06	7.3E-07	4.8E-07
2	1.5E-04	2.2E-06	5.7E-07	2.4E-07	1.2E-07	6.6E-08	3.9E-08	2.3E-08
3	3.8E-05	1.9E-06	5.4E-07	2.3E-07	1.2E-07	6.5E-08	3.9E-08	2.3E-08
4	1.7E-05	1.5E-06	5.1E-07	2.3E-07	1.2E-07	6.5E-08	3.9E-08	2.4E-08
5	9.4E-06	1.2E-06	4.6E-07	2.2E-07	1.1E-07	6.4E-08	3.8E-08	2.3E-08
6	6.0E-06	9.2E-07	4.1E-07	2.0E-07	1.1E-07	6.2E-08	3.7E-08	2.3E-08
7	4.1E-06	7.2E-07	3.6E-07	1.9E-07	1.0E-07	6.1E-08	3.7E-08	2.3E-08
8	2.9E-06	5.7E-07	3.2E-07	1.8E-07	1.0E-07	5.9E-08	3.6E-08	2.2E-08

In Figs. 3, 4 the isotherms are drawn at different *z* coordinates for only a quarter of the cross-section.



$z = 0.2$ m $z = 0.8$ m

Fig. 3 – Isotherms for 16 kA ampere-turn and a speed of 2 mm/s.



$z = 0.2$ m $z = 0.8$ m

Fig. 4 – Isotherms for 32 kA ampere-turn and a speed of 8 mm/s.

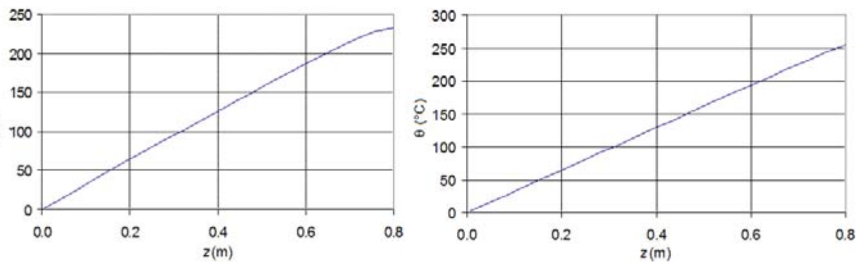
If the ampere-turn grows by a factor of r, the losses grow by approximately r^2 times and we can also increase the speed by r^2 times. Obviously the temperatures will grow more than double given the losses through the lateral part of the beam are smaller due to the shorter time of the beam inside the coil. It is, however, to be expected that the distribution of the temperature will loose its uniformity. In Fig. 4 the isotherms for a double ampere-turn $NI = 32$ kA and a four times bigger speed v = 8 mm/s are drawn.

5. ONE-DIMENSIONAL TEMPERATURE MODEL

From Figs. 3 and 4 we can see that in a cross-section of the beam the temperature is getting close to an uniform distribution. An extremely large reduction of the execution time can be accomplished if we assume the temperature depends only on the z coordinate and we obtain the equation

$$-\lambda S \frac{\mathrm{d}^2\theta}{\mathrm{d}z^2} + cVS \frac{\mathrm{d}\theta}{\mathrm{d}z} + \tau L\theta = P \, , \tag{22}$$

where (considering the first quadrant) $S = ab$ is the area of the cross-section, $L = a + b$ is the length of the portion of the boundary of the cross section through which the transfer of heat to the exterior is made and P is the global losses per unit of length computed in [1]: $p = \frac{1}{\sigma}\left(\frac{NI}{l}\right)^2 \sum_k \frac{\lambda_k^2 \widetilde{\Phi}_k^2}{1+\left(\frac{\lambda_k^2}{\omega\mu\sigma}\right)^2}$. The boundary condition at the entry of the beam inside the coil and at the exit are those given by (4) and (5).



a) NI = 16 kA, v = 2 mm/s, b) NI = 32 kA, v = 8 mm/s

Fig. 5 – The temperature along the beam.

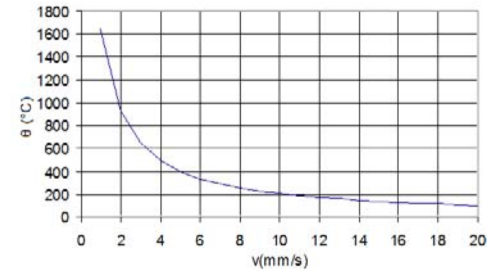


Fig. 6 – The final temperature of the beam in relation with the speed for an amper-turn of 32 kA.

We denote $Q=\dfrac{cV}{\lambda}$ and $\Gamma^2=\dfrac{\tau L}{\lambda S}$ and equation (22) can be written as $\dfrac{\mathrm{d}^2\theta}{\mathrm{d}z^2}-Q\dfrac{\mathrm{d}\theta}{\mathrm{d}z}-\Gamma^2\theta=-\dfrac{P}{\lambda S}$ with a similar form with equation (9) of the components of the series expansion and with the same boundary conditions. It has the solution

$$\theta=\frac{P}{\lambda S\Gamma^2}\left[1-\mathrm{e}^{-\frac{\Gamma^2 z}{Q''+Q'}}\left(1+\frac{\Gamma^2}{(Q''+Q')^2}\mathrm{e}^{-2Q''(l-z)}\right)\Big/\left(1+\frac{\Gamma^2}{(Q''+Q')^2}\mathrm{e}^{-2Q''l}\right)\right], \quad (23)$$

where $Q'=Q/2$ and $Q''=\sqrt{Q'^2+\Gamma^2}$. In Fig. 5 the temperature along the beam is presented for an ampere-turn of $NI=16\,\text{kA}$ and a speed $v=2$ mm/s and an ampere-turn of $NI=32\,\text{kA}$ and a speed v = 8 mm/s.

Determining the speed for a given ampere-turn in order to obtain a required temperature at the exit of the beam for the coil can be accomplished with the help of Fig. 6.

5. **CONCLUSIONS**

The solution for the temperature field distribution inside the moving beam has an analytical form as an expression of a series of spatial eigenfunctions of the cross section Ω_{xoy}. The series is rapidly convergent. In order to obtain the eigenvalues a transcendental equation is efficiently solved, by using a very fast fixed point technique. Once calculated they are valid for any point in the analyzed domain. To increase even further the computational speed, the matrices coefficients $Z_k(z)$ from series (7) can be computed for the required z coordinate and can be multiplied with the eigenfunctions Ψ_k, regardless of the (x, y) coordinates. The sizes of these matrices are small resulting from the small number of terms that must be summed in (7).

The paper also proposes a simplified procedure for determining the temperatures along the beam that can be used when the temperature field has small changes in the Ω_{xoy} plane. The procedure is very fast by not requiring determining the eigenvalues and eigenfunctions. The temperature in beams with non-rectangular cross-sections, where the calculation of the eigenfunctions and eigenvalues is particularly difficult, can be easily determined. The procedure allows a fast analysis of the beam's temperature in relation with the input parameters being, thus, an efficient instrument for optimizing these parameters.

Received on October 21, 2013

REFERENCES

1. M. Maricaru, M. Vasilescu, T. Leuca, I. F. Hănțilă, B. C. Stanciu, *Electromagnetic field computation in conducting beams using series expansion of space eigenfunctions,* Rev. Roum. Sci. Techn. – Électrotechn. et Énerg, **59**, *1*, 2014.
2. M. Vasiliu, *Nonsinusoidal two-dimensional eddy currents in a rectangular bar*, Rev. Roum. Sci. Techn. – Électrotechn. et Énerg., **36**, *3*, pp. 315–324, 1991.
3. M. Vasiliu, *Eddy current core losses with trapezoidal magnetization*, Rev. Roum. Sci. Techn. – Électrotechn. et Énerg., **37**, *1*, pp. 33–35, 1992.
4. M. Vasiliu, I.F. Hantila, *Eigenfunction expansion in a non-sinusoidal magnetic shielding problem*, Rev. Roum. Sci. Techn. – Électrotechn. et Énerg., **38**, *2*, pp. 199–206, 1993.
5. Istratescu V.I., *Introducere în teoria punctelor fixe*, Edit. Academiei RSR, Bucarest, 1973.