AN ITERATIVE FINITE ELEMENT – BOUNDARY ELEMENT METHOD FOR EFFICIENT MAGNETIC FIELD COMPUTATION IN TRANSFORMERS

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The paper proposes a new procedure for magnetic field computation in multiply connected ferromagnetic domains. The magnetic field in ferromagnetic bodies is computed using the Finite Element Method (FEM) with Dirichlet boundary conditions and the vector potential normal derivative is obtained. This becomes the boundary condition for the Boundary Element Method (BEM) applied in free space, from which the new value of the vector potential results. The relative high permeability of the ferromagnetic bodies insures the convergence of the procedure. The nonlinearity of the ferromagnetic bodies is treated using the polarization fixed point method (PFPM), in which the polarization is corrected in terms of magnetic field strength \( H \), in order to have a high permeability. The method was applied for the magnetic field computation in a transformer.

1. INTRODUCTION

The major disadvantages of FEM are: the usage of the artificial boundaries, which limit the computation domain, the difficult matching to structures containing moving media, the spurious forces between the mesh’s sub-domains. All of these were challenges for integral and hybrid methods development [1]. The hybrid FEM-BEM method consists of applying FEM for space discretization of the magnetic field equations in the ferromagnetic domains, the boundary condition being obtained by using BEM for the outer region, in the free space. Often, one prefers the formulation in \( H \) which permits the use of scalar potential in free space.

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[2], even if the ferromagnetic domains are multiply-connected [3]. The high magnetic permeability differences may lead to errors of the numerical solution.

Taking into account the number of unknowns the vector potential is often preferred in 2D problems. The hybrid FEM-BEM method preserves some advantages of FEM matrix, but, by adding BEM equations, the structure of the equation system matrix becomes worse, being sparse and asymmetrical. For multiply connected domains from 2D structures, the integral equation for inner areas with air does not allow the unique determination of the potential on the boundary [4], its value depending on a constant.

In this paper we present an iterative procedure which can be efficiently applied for structures with multiply-connected ferromagnetic domains. Admitting as known the value of the vector potential $A$ on the ferromagnetic domain boundary (Dirichlet) one uses FEM for magnetic field problem computation, obtaining the normal derivative of the vector potential. Then, taking into account the continuity of the tangential component of the magnetic field, by applying BEM, we obtain the new value of the potential on the boundary. For multiply connected domains (e.g., transformers), we propose a procedure for obtaining the vector potential on the inner region boundary. The proposed procedure has all the advantages of the FEM with Dirichlet boundary conditions.

2. THE MAGNETIC FIELD IN LINEAR FERROMAGNETIC BODIES

Supposing a null imposed current density, the vector potential equation $A$ in $\Omega$ domain is:

$$-\nabla \cdot \frac{1}{\mu} \nabla A = k \cdot (\nabla \times \frac{1}{\mu} I),$$

where $k$ is the unit vector of $oz$-axis, $\mu$ is the magnetic permeability and $I$ has the nature of a magnetic polarization and it is necessary to treat the nonlinear media. Suppose we know the value of $A$ on $\partial \Omega$ boundary of $\Omega$ domain (Dirichlet condition). We write:

$$A = \sum_{i \in \{N\}} A_i \varphi_i + \sum_{i \in \{NB\}} A_i \varphi_i,$$

where $\{N\}$ and $\{NB\}$ represents the sets of nodes from the inner domain and, respectively, from its boundary, $\varphi_i$ is the first order nodal element, in node $i$, and $A_j$, the value of the potential $A$ in that node. The FEM equations are:

$$\sum_{i \in \{N\}} a_{ji} A_i = - \sum_{i \in \{NB\}} a_{ji} A_i + t_j, \ j \in \{N\},$$
where:
\[
\begin{align*}
\alpha_i &= \int_{\Omega} \frac{1}{\mu} \nabla \phi_i \cdot \nabla \phi_j \, dS, \\
t_j &= \int_{\Omega} \left( \nabla \phi_j \times k \right) \cdot \frac{1}{\mu} \mathbf{I} \, dS.
\end{align*}
\] (4)

The matrix of \( a_{ji} \) entries from system (3) is sparse, symmetric and positive definite.

The storage effort is proportional to \( N \) and the system computation can be done very fast by using the sparse matrix techniques (conjugate gradient, Gauss-Seidel). By solving the system given by (3), one obtains the \( A_i \) potentials and then the tangential components of \( H \) on the segments from the boundary:

\[
H_{ij} = (\partial A_i / \partial n - I_j) / \mu_r.
\]

3. THE MAGNETIC FIELD IN FREE SPACE

For the beginning, we admit that the ferromagnetic domains are simply connected. The integral equation on \( \partial \Omega \) boundary in free space \( \Omega_0 \) is

\[
\lambda A(r) = \oint_{\partial \Omega} \left( \frac{R \cdot n}{R^2} A(r') \right) dl' - \oint_{\partial \Omega} \frac{1}{R} \ln \left( \frac{\partial A(r')}{\partial n} \right) dl' + A_0(r),
\] (5)

where: \( \chi \) is the angle under which a small neighbourhood of \( \Omega_0 \) is seen from the observation point; \( r, r' \) are the position vectors of the observation and integration points; \( R = r - r' \); \( n \) is the unit vector of the normal outside the ferromagnetic domain \( \Omega \); \( A_0 \) is the vector potential given by external sources.

![Fig. 1 – The notations for \( \Omega_0 \) domain and its boundary.](image1)

![Fig. 2 – The notations used for \( v_{k,q} \) and \( u_{k,q} \) entries computation.](image2)
By adopting a FEM mesh with triangular elements, $\partial \Omega$ boundary is approximated by a polygonal curve. We admit that on each segment on the boundary, the potential $A$ has a linear variation between the nodes from the ends of the segment and the derivative with respect to the normal is constant.

4. TRANSFORMER’S MULTIPLY CONNECTED DOMAIN

For simplicity, we choose the magnetic circuit of an unloaded functioning single-phase transformer in which we draw only one winding (Fig. 3). The multiply-connected domain of the transformer’s magnetic circuit defines two air areas, the outer and the inner windows. By placing the observation point in each of the $N^e$ nodes of the outer boundary $\partial \Omega^e$, equation (5) yields

$$\Lambda^eA^e = V^eA^e + U^e \frac{\partial A^e}{\partial n} + C^e \theta^e,$$

(6)

where $A^e$ and $\frac{\partial A^e}{\partial n}$ are vectors with $N^e$ entries of the nodes’ potentials on the boundary $\partial \Omega^e$ and of the derivatives with respect to the normal from the boundary’s segments, $\theta^e$ is the winding from the outer domain, $\Lambda^e$ is the diagonal matrix of the angles $\chi^e$, $V^e$ and $U^e$ are $N^e \times N^e$ matrices (see Fig.2)

$$v_{k,q} = \alpha(r_k, r_q, r_q^{II}, r_q^{III}), \quad u_{k,q} = \beta(r_k, r_q, r_q^{II}),$$

(7)

$$\alpha(r_k, r_q, r_q^{II}, r_q^{III}) = \int_{l_q} \frac{(R \cdot n)}{R^2} \frac{|r^e - r_q^e|}{|r_q^{II} - r_q^e|} dl + \int_{l_q^{III}} \frac{(R \cdot n)}{R^2} \frac{|r^e - r_q^{III}|}{|r_q^{III} - r_q^e|} dl,$$

(8)
The entries of $C^ε$ are

$$c_k^ε = \frac{\mu_0}{2\pi S^ε} \int_{\Omega^ε} \ln \left| \frac{r_k - r'}{r_k - r''} \right| \, ds', \quad \alpha_j^f$$

where $\Omega_j$ is the outer domain, with imposed current density, $S^ε$ is its area and $r'$ is the position vector of the integration point. Entries $q_k^ε, q_k^u, c_k^ε$ can be analytically computed. From relation (6) results

$$A^ε = Z^ε \frac{\partial A^T}{\partial n} + T^ε \theta^ε,$$

where $Z^ε = (\Lambda^ε - V^ε)^{-1}U^ε$ and $T^ε = (\Lambda^ε - V^ε)^{-1}C^ε$.

By placing the observation point in $N_i^1 = N_i - 1$ nodes of the inner window boundary $\partial \Omega_i$, where $N_i$ is the node’s number of this window. We admit that the potential of the $N_i$ is null. Alike relation (6), we obtain

$$A_i^1 = V_i A_i + U_i^T \frac{\partial A_i^T}{\partial n} + C_i^T \theta_i,$$

where $A_i$ and $C_i$ have, in this case, $N_i^1$ entries, $\Lambda_i$ is the diagonal matrix of $N_i^1 \times N_i^1$ of $\chi_i$ angles, $V_i$ is an $N_i^1 \times N_i^1$ matrix, and $U^ε$ is a matrix with $N_i^1 \times 1$ matrix. Alike equation (11), we obtain

$$A_i^1 = Z_i \frac{\partial A_i^T}{\partial n} + T_i \theta_i.$$  

In our example, $\theta^ε = - \theta_i$.

5. ITERATIVE PROCEDURE

In the neighbourhood of $\partial \Omega$ surface, the value of the potential $A$ and the tangential component of $H$ are preserved.

$$\partial A_i^T / \partial n = (\partial A / \partial n - I_{i}) / \mu_{r},$$
where \( I \) is the tangential component of the polarization. The bigger the value of the relative permeability \( \mu_r \) is, the more powerful the contraction given by relation (14) for the derivatives of the potential with respect to the normal is. As a consequence, one can propose the following iterative procedure:

a) Knowing the value of \( \partial A^e / \partial n \) on \( \partial \Omega^e \) boundary’s edges one obtains \( A^e \) with relation (11) (BEM).

b) Using the relation (14), from \( \partial A^i / \partial n \) value, one obtains \( A^i \) in the \( \partial \Omega^i \) boundary’s nodes, with the approximation of a constant (BEM).

c) Knowing \( A^e \) and \( A^i \), one determines \( A^r \) in the ferromagnetic domains.

This potential verifies the equation (3)

\[
\sum_{k \in \{n\}} p_{jk} A^r_k = - \sum_{k \in \{n'\}} p_{jk} A^e_k - \sum_{k \in \{n'\}} a_{jk} A^i_k + b_j, \quad j \in \{n\}. \tag{15}
\]

Then, one obtains \( \partial A^w / \partial n \) and \( \partial A^w / \partial n \) on \( \partial \Omega^e \) and \( \partial \Omega^i \) boundaries (FEM). In general, the line integral of \( \partial A^w / \partial n \) on \( \partial \Omega^i \) does not verify the Ampere theorem

\[
- \frac{1}{\mu_0} \oint_{\partial \Omega^i} \frac{\partial A^w}{\partial n} \, dl = 0^w \neq 0^i. \tag{16}
\]

Suppose potential \( A^w \) which has zero values on \( \partial \Omega^e \) boundary and unit value on \( \partial \Omega^i \) boundary (\( A^w_k = 1, \ k \in \{n'\} \)), the polarizations being zero (\( b_j = 0 \)). This potential verifies the equation (3)

\[
\sum_{k \in \{n\}} p_{jk} A^w_k = - \sum_{k \in \{n'\}} a_{jk}, \quad j \in \{n\}. \tag{17}
\]

To this potential corresponds the derivatives with respect to the normal \( \partial A^w / \partial n \) and \( \partial A^w / \partial n \) on \( \partial \Omega^e \) and \( \partial \Omega^i \) boundaries, having on \( \partial \Omega^i \) the line integral

\[
- \frac{1}{\mu_0} \oint_{\partial \Omega^i} \frac{\partial A^w}{\partial n} \, dl = 0^w. \tag{18}
\]

The potential \( A^w \) and the values \( \partial A^w / \partial n, \partial A^w / \partial n, \theta^w \) are computed only once, at the beginning of the proposed procedure. The correct value of the inner winding is obtained by obtaining the constant \( a^w \) such that
\[ \theta^{\prime} + c^{\prime} \theta^{\prime} = 0. \]  

(19)

The correct values of the potential from the ferromagnetic domain and the derivatives with respect to the normal are

\[ A = A^{\prime} + c^{\prime} A^{\prime}, \quad \frac{\partial A^{\prime}}{\partial n} + c^{\prime} \frac{\partial A^{\prime}}{\partial n} = \frac{\partial A^{\prime}}{\partial n} + c^{\prime} \frac{\partial A^{\prime}}{\partial n}. \]  

(20)

Steps a), b), c), d) are repeated until the difference between the values of \( \frac{\partial A}{\partial n} \) at two successive iterations verifies an error condition. \( Z \) and \( T \) matrices are computed only once, before the beginning of the iterations.

6. IMPOSED TERMINAL VOLTAGE

The value of the potential \( A_0^e \) in an arbitrary point from \( \Omega^e \) is obtained using the relation (6), in which \( \chi = 2\pi \), and the position vector \( r \) from relations (7) and (9) corresponds to the observation point inside \( \Omega^e \).

\[ A_0^e = \frac{1}{2\pi} (V_0^e A^e + U_0^e \frac{\partial A^e}{\partial n} + C_0^e \theta^e). \]  

(21)

Replacing \( A^e \) from (11), we have

\[ A_0^e = Z_0^e \frac{\partial A^e}{\partial n} + T_0^e \theta^e, \]  

(22)

where \( Z_0^e = \frac{1}{2\pi} V_0^e Z^e + U_0^e, \) \( T_0^e = \frac{1}{2\pi} V_0^e T^e + C_0^e \). Integrating the relation (22) on \( \Omega^j \) domain, of area \( S^e \), we obtain the contribution of the outer winding to the beam flux

\[ \Phi^j = \frac{1}{S} \int_{\Omega^j} A_0^e d\mathbf{x} + c_0 = Z_0^e \frac{\partial A^e}{\partial n} + T_0^e \theta^e - T_0^e \int_{\Omega^j} d\mathbf{l} + c_0. \]  

(23)

In the same way one obtains the contribution of the inner winding to the beam flux

\[ \Phi^i = \frac{1}{S} \int_{\Omega^i} A_0^i d\mathbf{x} + c_0 = Z_0^i \frac{\partial A^i}{\partial n} - T_0^i \int_{\Omega^i} d\mathbf{l} + c_0. \]  

(24)
where the \( c_0 \) constant results from the fact that in \( \Omega^{i} \) the potential is obtained with the approximation of a constant. So, the average beam flux of the winding is

\[
\Phi = \Phi^{i}_j - \Phi^{e}_j = Z_\Phi^{i} \frac{\partial A^{i}}{\partial n} - T_\Phi^{i} \int_{\partial \Omega^{i}} \frac{\partial A^{i}}{\partial n} \, dl + c_0 - Z_\Phi^{e} \frac{\partial A^{e}}{\partial n} + T_\Phi^{e} \int_{\partial \Omega^{e}} \frac{\partial A^{e}}{\partial n} \, dl.
\] (25)

The matrices \( Z_\Phi^{i}, Z_\Phi^{e} \) and the constants \( T_\Phi^{i}, T_\Phi^{e} \) are computed only once, before the beginning of the iterative procedure.

If we admit that the resistive voltage drop on the winding can be neglected (since we chose the unloaded operation), then the average beam flux is a primitive of the terminal voltage: \( \Phi = \frac{1}{2\pi f N_{sp}} \int_{\Omega} \Phi \, d\Omega \), where \( N_{sp} \) is the number of turns and \( f \) is the frequency. In our paper, for the sinusoidal voltage we choose the sinusoidal primitive. The iterative procedure is modified as follows:

a) Knowing the value of \( \frac{\partial A^{i}}{\partial n} \) on edges of \( \partial \Omega^{i} \) boundary, one obtains \( A^{e} \) using relation (11) (BEM).

b) Using relation (13), from \( \frac{\partial A^{i}}{\partial n} \) value one obtains \( A^{e} \) in the nodes of \( \partial \Omega^{i} \) boundary with the approximation of a constant \( c_0 \) (BEM).

c) One determines \( c_0 \) from relation (25) and is added to \( A^{e} \), obtaining: \( A^{i} = A^{e} + c_0 \).

d) \( A \) is determined by solving FEM ferromagnetic domain problem.

7. NONLINEAR MEDIA

The field problem in nonlinear media is solved using MPFP [5]. The nonlinear constitutive relation \( B = \mathbf{F}(\mathbf{H}) \) is replaced by

\[
B = \mu H + I,
\] (26)

where the additive term \( I \), of a magnetic polarization nature, is corrected as a function of \( H \) by

\[
I = \mathbf{F}(\mathbf{H}) - \mu H = G(\mathbf{H}),
\] (27)

Having the fictitious polarization \( I \), one determines the magnetic field \( (B, H) \) which has the linear constitutive relation (26), following the above presented iterative procedures. Then \( I \) is updated using relation (27). The two steps are repeated until the difference norm between the polarizations of two successive
iterations is sufficiently small \(\left\| I^{(n)} - I^{(n-1)} \right\|^2 = \int_{\Omega} \frac{1}{\mu} \left(I^{(n)} - I^{(n-1)}\right)^2 dS < \varepsilon_f^2\). The fictitious magnetic permeability \(\mu\) can be chosen such that \(G\) is a contraction. For example, for isotropic media, where \(I = f(H) - \mu H = g(H)\) and \(I = \frac{H}{H}\), it is necessary that \(\mu > \frac{(dg/dH)_{\max}}{2}\). This condition is a convenient one because it insures the convergence of the linear field problem iterative presented above.

8. EXAMPLE

We consider the transformer from Fig. 3. The domain occupied by the ferromagnetic media is multiply-connected. We admit that the transformer is working unloaded and the input voltage is a sinusoidal one, with a maximum value of 329.5V, the primary winding having 325 turns. The thickness of the transformer is 48 mm. Because the magnetic flux density enters the saturation area, it has been chosen a smaller computation permeability \(\mu = \frac{5\mu_{\max} + 4\mu_{\min}}{9} > \frac{\mu_{\max}}{2}\), where \(\mu_{\min} = 12\mu_0\) and \(\mu_{\max} = 9947\mu_0\). In Fig. 4 we present the plot of the magnetization current during half-period.

![Fig. 4 – Magnetization current during half-period.](image-url)
9. CONCLUSIONS

The main contribution of this paper is the iterative procedure for solving the equations of FEM-BEM hybrid method in multiply connected ferromagnetic bodies (transformers). The fast convergence of this procedure is due to the high difference between the magnetic permeability in air and in ferromagnetic bodies. Using FEM for solving the problem in ferromagnetic bodies with Dirichlet boundary conditions, we have a symmetric, sparse and positive definite system’s matrix. The correction of the boundary condition is performed by BEM. The proposed procedure in this paper has all FEM advantages. For highly saturated media, the number of iterations in PFPM procedure can increase sometimes up to 87, while for the hybrid procedure 3 iterations were sufficient; in most cases just one iteration being sufficient. FEM mesh had 1309 nodes and 2370 triangles. To analyze 48 cycles it took 216 s for a notebook with a 2.5 GHz processor.

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