AN EXTENSION OF THE MINIMUM ENERGY PRINCIPLE IN STATIONARY REGIME FOR ELECTRIC AND MAGNETIC CIRCUITS

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The paper describes the authors’ further preoccupation towards an extension of the minimum energy principle in stationary regime established for the d.c. and a.c. circuits, to the linear magnetic circuits and to the electric circuit with capacitors. It is tried to give an interpretation to the results obtained from the minimization of an energy functional corresponding to the circuit under study.

1. INTRODUCTION

Conservative systems accept the definition of functionals expressed in terms of power or energy. Calculating the limits of these functionals represents an important breakthrough in formulating and solving optimization problems. The steady states in the mechanic, thermal, or electric conservative systems generally represent limit states from an energy point of view. For example, in the classical mechanics, Hamilton’s principle [1], which refers to the state of minimal action in time-related evolution of system, from one quasi-stationary state to another, along the curve \( \gamma : [a, b] \to \mathbb{R} \), is defined by the minimum of the functional

\[
F = \int_{a}^{b} (T - U) \, dt ,
\]

also called the „action integral”, where \( F \) is the energy functional, \( T \) is the kinetic energy (so-called “life power”) of the system, and \( U \) is the mechanic effect of the power system which works on the system under consideration.

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In the theory of the electric circuits, the results obtained by Miller and Stern [2, 3] in connection with the "co-content" function for the non-linear resistive and reciprocal circuits, have a notable theoretical importance. Others valuable contributions have appeared in the literature, C. A. Desoer and E. Kuh [4] (pp. 770-772), and W.E Smith [5, 6], each shedding more light on some aspects of the minimal dissipated power in the linear and resistive circuits.

In the Romanian technical literature, V. Ionescu [7], and C.I. Mocanu [8] (pp. 350-353) have brought important contributions to the theory of the minimum and maximum of the powers in the electric circuits. All of these results are basically consequences of Maxwell's principles of minimum-heat [9] (pp. 407-408).

Research works published by authors such as [10, 17] have thoroughly demonstrated the minimum energy principle for the electric circuits in stationary and quasi-stationary regime. Thus, for the d.c. circuits, this takes the form of the first principle of minimum absorbed power: "The minimum of the absorbed power by the branches of a linear and resistive circuit in a stationary regime is verified by the solutions of the currents and voltages in the circuit, and these are the currents and voltages that verifies the first and the second of Kirchhoff's theorems" or, in other words: "In a reciprocal and resistive d.c. circuit the currents and voltages get distributed such as the absorbed power by the branches of the circuit should be minimal".

In the case of a.c. circuits, the description is the second principle of the minimum active and reactive absorbed power: "The minimum of active and reactive absorbed (generated) power by the branches of a linear circuit in quasi-stationary a.c. is verified by the solutions in currents and voltages in the circuit, and these are the currents and voltages that verify the first and second Kirchhoff's theorems". And here is just another description of the same principle: "In the linear and reciprocal circuit in quasi-stationary a.c. regime the currents and voltages are distributed such as the active and reactive absorbed (generated) power by the branches of the circuit should be minimal".

Starting from these principles and from the analogies that exist between the magnetic and electric circuits, the present work aims at demonstrating the minimum energy principle in the electric circuits with capacitors and in the linear magnetic circuits in stationary regime.

2. THE MINIMUM ENERGY PRINCIPLE FOR ELECTRIC CIRCUITS WITH CAPACITORS

The field literature includes valuable works [9: pp. 407-408, 18, 19, 20: pp. 130-138] which describe results concerning the minimum electrostatic energy
in a system of conductors charged with electric charges. We shall now carry out a
demonstration of the minimum energy principle in a circuit made of capacitors.

By applying Kirchhoff’s second theorem for a given branch of the network
(Fig. 1), connected between two nodes with potentials $V_x$ and $V_y$, respectively,
made of a capacitor $C$ charged with electric charge $q$ from the voltage source $E$,
we get the following expression of the voltage between the two nodes

$$U = V_x - V_y = U_C + U_E = \frac{q}{C} - E. \quad (2)$$

![Fig. 1 – Branch with capacitor.](image)

The electrostatic energy of the system expressed in terms of potentials is

$$W_e = \frac{1}{2} q (V_x - V_y + E) = \frac{1}{2} C (V_x - V_y + E)^2. \quad (3)$$

The electrostatic energy, having values obtained by computing the integral of
a vectorial function, is an energy functional $F_E$ defined within the vectorial space
of the electrical field and having values in the positive real set $\mathbb{R}^+$

$$F_E \equiv W_E = \frac{1}{2} C (V_x - V_y + E)^2. \quad (4)$$

The functional is positive defined, i.e. $F_E(V_x, V_y) \geq 0$, whatever the values of
the potentials $V_x$ and $V_y$ might be. This means that its extreme points will be
minimum points, and therefore the solutions of the system with partial derivatives

$$\frac{\partial F_E}{\partial V_x} = 0, \quad \frac{\partial F_E}{\partial V_y} = 0 \quad (5,6)$$

will correspond to the minimum electrostatic energy of the network. For example,
by computing the system (5) for all the $l_k$ branches connected the $x$ node, we will
obtain a relation that verifies Kirchhoff’s first theorem for a circuit with capacitors,
expressed in the $x$ node
\[
\sum_{l_{k}\in x} \frac{\partial F_{l_{k}}}{\partial V_{l_{k}}} = \sum_{l_{k}\in x} C_{l}(V_{l_{k}} - V_{l_{k}} + E_{l_{k}}) = \sum_{l_{k}\in x} q_{l_{k}} = 0. \tag{7}
\]

A similar equality is to be obtained through computation of the system (6).

We can now state the following principle (the minimum energy principle in circuits with capacitors): In the circuits in electrostatic regime the electric charges and the potentials of the capacitors, verify the Kirchhoff’s theorems, and correspond to the minimum electrostatic energy of the circuit.

3. THE MINIMUM ENERGETIC PRINCIPLE FOR LINEAR MAGNETIC CIRCUITS IN STATIONARY REGIME

The analogies between the electric and magnetic circuits as well as the similar definition of the electrostatic and magnetic energy have led to the statement of minimum energy principle of the linear magnetic circuits in stationary regime.

If we consider a coil having \( N \) turns, crossed by the conduction current \( i \), situated on a linear, homogenous and isotropic magnetic material, of reluctance \( R_{m} \), which has the equivalent magnetic circuit shown in Fig. 2, then Kirchhoff’s second theorem for magnetic circuits in stationary regime is:

\[
U_{m} + \theta = R_{m} \Phi_{f} . \tag{8}
\]

![Fig. 2 – The equivalent magnetic circuit of a coil.](image)

If \( V_{m,x} \) and \( V_{m,y} \) are the magnetic potentials of the nodes to which this branch is connected, then the magnetic flux \( \Phi_{f} \) can be expressed in terms of these and of the current linkage (\( \theta = Ni \))

\[
\Phi_{f} = \frac{V_{m,x} - V_{m,y} + \theta}{R_{m}} . \tag{9}
\]

The magnetic energy of the circuit can be expressed as a dependency on the magnetic potentials
5 Minimum energy principle for circuits in stationary regime

\[ W_m = \frac{1}{2} R_m \Phi_j^2 = \frac{(V_{m,x} - V_{m,y} + \Theta)^2}{2R_m}. \]  (10)

Similarly to the case of the electrostatic energy, the magnetic energy, having values received from the computation of the integral of a vectorial function, is an energy functional \( F_M \) defined to the vectorial space of the magnetic field \( S_M \), with values in the real positive set \( \mathbb{R}^+ \). \( F_M : S_M \rightarrow \mathbb{R}^+ \)

\[ F_M = W_M = \frac{(V_{m,x} - V_{m,y} + \Theta)^2}{2R_m}. \]  (11)

The energy functional \( F_M \) is positively defined, that is \( F_M(V_{m,x}, V_{m,y}) \geq 0 \), whatever the values of the magnetic potentials \( V_{m,x} \) and \( V_{m,y} \) might be. Consequently, its extreme points will be minimum points and therefore the solutions of the systems with partial derivatives

\[ \frac{\partial F_M}{\partial V_{m,x}} = 0, \quad \text{and} \quad \frac{\partial F_M}{\partial V_{m,y}} = 0 \]  (12,13)

will correspond to the minimum magnetic energy of the circuit.

For instance, by calculating system (12) for all the branches \( l_k \) with coils, connected to the node \( x \), we obtain a relation that verifies Kirchhoff’s first theorem for magnetic circuits expressed in node \( x \)

\[ \sum_{l_k \in x} \frac{\partial F_{m,k}}{\partial V_{m,x}} = \sum_{l_k \in x} \left( \frac{V_{m,x} - V_{m,y} + \Theta_k}{R_{m_k}} \right) = \sum_{l_k \in x} \Phi_i = 0. \]  (14)

A similar relation is obtained considering system (13).

We can state the following principle (the minimum energy principle for linear magnetic circuits in stationary regime): In the linear magnetic circuits in stationary regime, the linking fluxes and the magnetic potentials verify the Kirchhoff’s first, respectively second theorem for magnetic circuits, and correspond to the minimum magnetic energy of the circuit.

4. EXAMPLES

4.1. CIRCUIT WITH CAPACITORS

We consider the circuit with three capacitors, not charged initially, connected like in Fig. 3. By applying Kirchhoff’s second theorem we can express the electrostatic energy \( W_E \) of the system in terms of potentials \( V_x \) and \( V_y \)
\[
W_E = \frac{1}{2} \left[ q_1 U_1 + q_2 U_2 + q_3 U_3 \right] = \frac{1}{2} \left[ C_1 (E - V_x + V_y) + C_2 (V_x - V_y) + C_3 (V_y - V_x) \right].
\] (15)

![Circuit with capacitors](image)

Fig. 3 – Circuit with capacitors.

The optimisation of the energy functional \( F_E \) as a dependency on the potential \( V_x \),

\[
\frac{\partial F_E}{\partial V_x} = \frac{\partial W_E}{\partial V_x} = 0,
\] (16)

leads to the verifying of Kirchhoff’s first theorem in node \( x \)

\[
\frac{\partial W_E}{\partial V_x} = C_1 (E - V_x + V_y) + C_2 (V_x - V_y) + C_3 (V_y - V_x) = -q_1 + q_2 + q_3 = 0.
\] (17)

In a similar way, if the optimisation of the energy functional of the circuit is performed relative to the potentials \( V_y \), Kirchhoff’s first theorem comes to be verified in node \( y \)

\[
\frac{\partial W_E}{\partial V_y} = C_1 (E - V_x + V_y) - C_2 (V_x - V_y) - C_3 (V_y - V_x) = q_1 - q_2 - q_3 = 0.
\] (18)

As a conclusion, the potentials and the electric charges of the circuit are thus distributed as to get a minimum electrostatic energy of the circuit.

### 4.2. MAGNETIC CIRCUIT WITH COILS

Let us consider a magnetic circuit that contains three coils, connected to the nodes \( x \) and \( y \), like in Fig. 4. By applying Kirchhoff’s second theorem for magnetic circuits, we can define the magnetic energy of the circuit \( W_m \) in terms of the magnetic potentials \( V_{m,x} \) and \( V_{m,y} \)
where $R_{m,1}, R_{m,2}, R_{m,3}$ are the magnetic reluctances of the three branches, while $\theta_1, \theta_2, \theta_3$ are the current linkages of the three coils.

The optimisation of the energetic functional of the circuit in relation to the magnetic potential $V_{m,x}$

$$\frac{\partial F_M}{\partial V_{m,x}} = \frac{\partial W_M}{\partial V_{m,x}} = 0,$$

leads to the verifying of Kirchhoff’s first theorem for magnetic circuit in node $x$

$$\frac{\partial W_M}{\partial V_{m,x}} = \frac{V_{m,x} - V_{m,y} + \theta_1}{R_{m,1}} + \frac{V_{m,x} - V_{m,y} + \theta_2}{R_{m,2}} + \frac{V_{m,x} - V_{m,y} + \theta_3}{R_{m,3}} =$$

$$= -\Phi_{f,1} + \Phi_{f,2} + \Phi_{f,3} = 0.$$

In a similar way, if the optimisation of the energy functional of the circuit is performed in relation to the potential $V_y$, Kirchhoff’s first theorem for magnetic circuit comes to be verified in node $y$

$$\frac{\partial W_M}{\partial V_{m,y}} = \frac{V_{m,y} - V_{m,x} + \theta_1}{R_{m,1}} - \frac{V_{m,x} - V_{m,y} + \theta_2}{R_{m,2}} - \frac{V_{m,x} - V_{m,y} + \theta_3}{R_{m,3}} =$$

$$= \Phi_{f,1} - \Phi_{f,2} - \Phi_{f,3} = 0.$$
In conclusion, the magnetic potentials and the fluxes linking the coils in the circuit get such distribution as to allow the magnetic energy in the circuit to be minimal.

5. CONCLUSIONS

By applying the energy functional in the analysis of electrical and magnetic circuits, it is possible to assess the equilibrium state, from an energy point of view, existing in the system at different moments of time.

Practically, to reach a balanced and stable state in the circuits, the energy in the circuit must be minimal, providing that the state variables of the circuit should verify Kirchhoff’s theorems. The formulation of the energy functionals in the Hilbert space and the determining of their extremes lead to the conclusion that the equilibrium states in the circuits represent a minimum from an energy point of view.

For this reason, the determining of the functionals in linear circuits is a very important problem, didactically and theoretically, as well as for its practical applications.

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