A COMPREHENSIVE ANALYSIS OF THE SCALAR COMPANDOR MODEL DESIGNED USING SPLINE FUNCTIONS

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Key words: Optimal compressor functions, Companding technique, Approximative spline functions.

This paper discusses the design of the new scalar compandor model using spline functions of the first and the second degree. The spline functions are used for the approximation of the optimal compressor function. Based on approximative spline functions, for the Gaussian source at the input of the quantizer, designing of the scalar compandor model is done. The signal to quantization noise ratio value which is achieved by compandor, designed in this way, is very close to the signal to quantization noise ratio value of the optimal compandor, where the design complexity of the proposed model is significantly smaller.

1. INTRODUCTION

Quantization is the process in which the continuous range of value of an analog signal is sampled and divided into non-overlapping (but not necessarily equal) subranges, and a discrete, unique value is assigned to each subrange. The independent quantization of each signal value or parameter is termed scalar quantization. A device that performs quantization is called a quantizer [1, 2]. When all the quantization intervals are of the same width, the quantizer is uniform. Otherwise, the quantizer is nonuniform (different width of the quantization intervals). Uniform quantizers are usually applied as analog-to-digital converters (ADC) [1, 2], while the nonuniform quantizers improve signal quality or reduce quantization rate [1–3].

Nonuniform quantizers are based on passing the signal via a compressor block before performing regular uniform quantization. At the receiver side, the

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Rev. Roum. Sci. Techn. – Électrotechn. et Énerg., **60**, *3*, p. 283–292, Bucarest, 2015

signal is passed through an expander block [1, 2]. The electronic circuit consisting of a compressor, an uniform quantizer, and an expandor in cascade is called a compandor [1, 2].

Quantizer is completely defined if nonlinear compressor function is known. However, it is well known that designing the optimal nonlinear companding quantizers for a Gaussian source is very complex due to the difficulties in determining the inverse optimal compressor function [1, 4–6]. In order to solve this problem, that is to simplify determining the inverse optimal compressor function, the approximation the optimal compressor function by other function is often done. This way the quantizer design is considerably simplified [7, 8].

By applying the similar methodology as in [9], in this paper we propose design of companding quantizer of a Gaussian source by approximating the optimal compressor function using the first-degree spline functions and the second-degree spline functions. The support region of companding quantizer is divided into $2L = 4$ equal segments. The number of cells within segments is different. With such quantizer design, the value of the signal to quantization noise ratio (SQNR) that is very close to the value of SQNR of the optimal compandor is achieved.

The remainder of the paper is organized as follows: In Section 2 the detailed description of determining spline functions, as well as their application in approximation of the optimal compressor function for Gaussian source are given. Scalar compandor design based on optimal compressor function approximating by spline functions is described in Section 3. Section 4 presents a discussion of numerical results obtained for the SQNR of the proposed companding quantizer. Finally, Section 5 gives a conclusion.

2. DETERMINING SPLINE FUNCTIONS

The optimal nonlinear compressor function $c(x):[-x_{\max}, x_{\max}] \to [-x_{\max}, x_{\max}]$, which gives the maximum SQNR for the reference variance σ^2 of an input signal x is defined as [1, 2]:

$$c(x) = \begin{cases} x_{\max} \dfrac{\int_0^{|x|} p^{1/3}(t)\mathrm{d}t}{\int_0^{x_{\max}} p^{1/3}(t)\mathrm{d}t} \operatorname{sgn}(x), & |x| \le x_{\max}, \end{cases} \tag{1}$$

where $x_{\max}$ denotes the support region threshold of the optimal companding quantizer and $p(t)$ is a symmetric Gaussian probability density function (PDF), which is defined as follows [1]:

$$p(t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{t^2}{2\sigma^2} \right). \tag{2}$$

By substituting (2) into (1) the well known form of the optimal compressor function of a Gaussian source is obtained:

$$c(x) = \left\{ x_{\max} \frac{1 - \mathrm{erfc}\left(\frac{|x|}{\sqrt{6}\sigma} \right)}{1 - \mathrm{erfc}\left(\frac{x_{\max}}{\sqrt{6}\sigma} \right)} \mathrm{sgn}(x), \quad |x| \le x_{\max} \right., \tag{3}$$

where $\mathrm{erfc}(u) \frac{2}{\sqrt{\pi}} \int_u^{\infty} \exp(-t^2) \mathrm{d}t$. Since $c(-x) = -c(x)$ holds, in continuation of the paper the approximation of the optimal compressor function is constrained on domain $[0, x_{\max}]$.

Generally, the simplest spline is composed of linear functions of the form $S_k(x) = a_k x + b_k$, for $x \in [x_k, x_{k+1}]$, i.e. a straight line between the two successive points x_k and x_{k+1}. The coefficients a_k and b_k are determined by the conditions $S_k(x_k) = f_k$ and $S_k(x_{k+1}) = f_{k+1}$, $k = 1, \ldots, n-1$. The interpolating function is continuous but it is not differentiable, i.e. not smooth, at the interior points: $S_k^{'}(x_{k+1}) \neq S_{k+1}^{'}(x_{k+1})$.

The form of the first-degree spline function, $z(x)$, which is used for approximation of the optimal nonlinear compressor function $c(x)$ in this paper, on domain $[0, x_{\max}]$, is [10]:

$$z_i(x) = c(x_i) + h_i(x - x_i),\ x \in [x_{i-1}, x_i],\ i = 1, \ldots, L, \tag{4}$$

where $x_0 = 0$ and number of segments $L = 2$. The coefficients of direction of the line, h_1 and h_2, are given by the formula:

$$h_i = \frac{c(x_i) - c(x_{i-1})}{x_i - x_{i-1}}, \quad i = 1, \ldots, L. \tag{5}$$

Consider the spline composed of $n-1$ quadratic polynomials of the form $S_k(x) = a_k x^2 + b_k x + c_k$, for $x \in [x_k, x_{k+1}]$, $k = 1, \ldots, n-1$. The second-degree spline function is a piecewise quadratic function that is continuous and has continuous derivative at the knots but not second derivative. The form of the approximate second-degree spline function, $w(x)$, which is used for approximation of the optimal nonlinear compressor function $c(x)$ in this paper, for the $L=2$ segments, is as follows [10]:

$$w_i(x)=p_i+q_i x+r_i x^2,\ x\in[x_{i-1},\ x_i],\ i=1,\ldots,N,\ x_0\le x\le x_{\max}, \quad (6)$$

where $x_0=0$. The coefficients of the approximate function $w(x)$ are determined by satisfying the governing continuity conditions on each spline thresholds. We need $3(n-1)$ equations to determine the $3(n-1)$ coefficients $\{a_k,b_k,c_k\}$. The conditions $S_k(x_k)=f_k$ and $S_k(x_{k+1})=f_{k+1}$ provide $2(n-1)$ equations while the continuity of the derivative at the interior points, $S_k^{'}(x_{k+1})=S_{k+1}^{'}(x_{k+1})$, $k=1,\ldots,n-1$, provides $n-2$ extra equations. By solving the following system of equations [10]:

$$w(0)=0, \quad (7)$$

$$c(x_1)=p_1+q_1x_1+r_1x_1^2, \quad (8)$$

$$c(x_1)=p_2+q_2x_1+r_2x_1^2, \quad (9)$$

$$c(x_2)=p_2+q_2x_2+r_2x_2^2, \quad (10)$$

$$\lim_{x\to x_1^-} w^{'}(x)=\lim_{x\to x_1^+} w^{'}(x)\Rightarrow q_1+2r_1x_1=q_2+2r_2x_1, \quad (11)$$

$$w^{'}(x_2)=0\Rightarrow q_2+2r_2x_2=0, \quad (12)$$

we obtain the values of coefficients: $p_1, q_1, r_1, p_2, q_2, r_2$.

3. DESCRIPTION OF THE COMPANDOR MODEL BASED ON SPLINE FUNCTIONS

In this section, a detailed description of the companding quantizer model whose design is based on spline functions is given. In the first case, the companding quantizer design is done by spline functions of the first degree. In the second case, the design is done using the second-degree spline functions. Spline functions of the first and second degree by which the optimal compressor function is approximated are used to reduce the complexity of designing the companding quantizer model with regard to the optimal compandor.

As described in [5], the support region threshold for the optimal companding quantizer, by which the companding quantizer design is done, equals to:

$$x_{\max}=\sigma\sqrt{6\ln N}\left[1-\frac{\ln\ln N}{4\ln N}-\frac{\ln\left(3\sqrt{\pi}\right)}{2\ln N}\right]. \quad (13)$$

For the support region threshold determined by the equation (13), the total number of the reproduction levels per segments is determined by the following expression:

$$\sum_{i=1}^{L}\frac{N_i}{2}=\frac{N-2}{2}. \tag{14}$$

From the condition that all reproduction levels are equidistant at compressor output, the number of reproduction levels per segments is determined:

$$\frac{N_i}{2}=\frac{N-2}{2}\frac{c(x_i)-c(x_{i-1})}{c(x_L)}, \quad i=1,\dots,L. \tag{15}$$

For the scalar compandor quantizer designed by the first-degree spline functions, the values of the reproduction levels are determined by the inverse first-degree spline functions as follows:

$$y_{i,j}=z_i^{-1}\left(\left(\frac{2j-1}{2}\right)\Delta\right), \quad i=1, \quad j=1,\dots,\frac{N_i}{2}, \tag{16}$$

$$y_{i,j}=z_i^{-1}\left(c(x_i)+\left(\frac{2j-1}{2}\right)\Delta\right), \quad i=2,\dots,L, \quad j=1,\dots,\frac{N_i}{2}. \tag{17}$$

For the value of reproduction levels is taken the solution that belongs to the spline function domain. The value of reproduction levels of the compandor designed by the second-degree spline functions is determined by the equations (16) and (17), except instead of calculating the inverse first-degree spline function $z^{-1}(x)$, the inverse second-degree spline function $w^{-1}(x)$ is calculated. In the remainder, determination of all parameters that characterize the proposed companding quantizer is done for the case of the first-degree spline function $z(x)$. Substituting the first-degree spline function $z(x)$ with the second-degree spline function $w(x)$, parameters that characterize companding quantizer designed by the second-degree spline functions is determined.

Step size at the output of compressor Δ is determined with:

$$\Delta=\frac{2x_{\max}}{N-2}, \tag{18}$$

while the cell length of the considered companding quantizer is determined by:

$$\Delta_{i,j}=\frac{\Delta}{z_i'\left(y_{i,j}\right)}, \quad i=1,\dots,L, \quad j=1,\dots,\frac{N_i}{2}. \tag{19}$$

The granular distortion for a companding quantizer is defined with Benett's integral [1,2], and it has the following form:

$$D_g=2\frac{x_{\max}^2}{3(N-2)^2}\sum_{i=1}^{L}\sum_{j=1}^{N_i/2}\frac{p(y_{i,j})}{\left[z_i'(y_{i,j})\right]^2}\Delta_{i,j}. \tag{20}$$

The overload distortion of companding quantizer can be expressed by [1, 2]:

$$D_o = 2\int_{x_{\max}}^{\infty}\left(x - y_{\max}\right)^2 p(x)\mathrm{d}x\,. \tag{20'}$$

The last reproduction level is denoted by $y_{\max}$. For companding quantizers proposed in this paper, the last reproduction level is determined by the centroid condition [1, 2]:

$$y_{\max} = \frac{\int_{x_{\max}}^{\infty} xp(x)\mathrm{d}x}{\int_{x_{\max}}^{\infty} p(x)\mathrm{d}x}\,. \tag{21}$$

As described in [5], combining the set of equations (13), (21) and (22), for the Gaussian PDF (2), we determine the overload distortion of the proposed companding quantizer:

$$D_o = \sqrt{\frac{2}{\pi}}\frac{1}{x_{\max}^3}\exp\left(-\frac{x_{\max}^2}{2}\right). \tag{22}$$

The sum of the granular D_g and the overload D_o distortion represents the total distortion. The performance of the proposed companding quantizer models are evaluated by the signal to quantization noise ratio [1, 2]:

$$\mathrm{SQNR} = 10\log\left(\frac{\sigma^2}{D_g + D_o}\right) = 10\log\left(\frac{\sigma^2}{D}\right). \tag{23}$$

Numerical results presented in next section are achieved for the reference input variance of $\sigma^2 = 1$.

4. NUMERICAL RESULTS AND DISCUSSION

Performances of the quantizers proposed in this paper are achieved for the case when the number of segments is equal to $2L = 4$ and for values of the number of levels $N = 16$, $N = 32$, $N = 64$, $N = 128$. Table 1 shows values of the segment threshold x_1, values of the support region threshold $x_2 = x_{\max}$, values of the optimal compressor functions $c(x_1)$ and $c(x_2)$ and values of the coefficient of direction of the line, h_1 and h_2, for a different number of levels N (from $N = 16$ to $N = 128$). Based on these parameters, the first-degree spline approximation is formed. The first-degree approximative spline functions described by (4) are achieved when parameters' values from the Table 1 are replaced in (4).

Table 1

The values of the segment thresholds, optimal compressor functions, and coefficients of direction of the line

N	x_1	x_2=x_{max}	$c(x_1)$	$c(x_2)$	h_1	h_2
16	1.2373	2.4746	1.5339	2.4746	1.2397	0.7603
32	1.5259	3.0519	2.0579	3.0519	1.3487	0.6514
64	1.7819	3.5638	2.5843	3.5638	1.4503	0.5497
128	2.0137	4.0274	3.1029	4.0274	1.5409	0.4591

The values of the coefficients which are determined by solving equation systems which is described by (7) – (12) are shown in Table 2. The second-degree approximative spline functions described by (6) are achieved when parameters' values from the Table 2 are replaced in (6).

Table 2

The values of the segment thresholds and coefficients for forming the second-degree spline functions

N	x_1	x_2=x_{max}	p_1	q_1	r_1	p_2	q_2	r_2
16	1.2373	2.4746	0	0.9588	0.2269	1.2882	3.0411	0.6144
32	1.5259	3.0519	0	1.3945	0.0301	0.9238	2.6054	0.4269
64	1.7819	3.5638	0	1.8012	0.1969	0.3542	2.1988	0.3085
128	2.0137	4.0274	0	2.1636	0.3092	0.3294	1.8364	0.2279

Approximation of the optimal compression function, $c(x)$, by the first-degree spline approximation, $z(x)$, and the second-degree spline approximation, $w(x)$, for the number of levels $N = 128$, is shown in Fig. 1. As it can be seen in Fig. 1, the optimal compressor function is very well approximated by the second-degree spline functions.

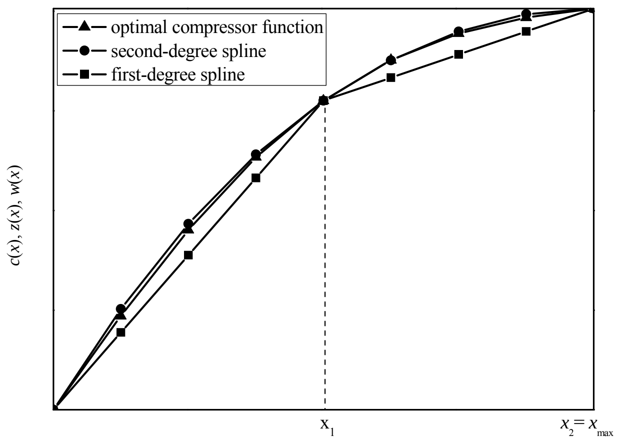

Fig. 1 – Spline functions of the first and second degree and the nonlinear optimal compressor function for the number of segments $2L = 4$ and the number of levels $N = 128$.

Table 3 shows the values of SQNR of the quantizer described in [4], ($SQNR^{[4]}$), the values of SQNR of the proposed companding quantizer based on the first-degree spline approximation, ($SQNR^{FD}$), the values of SQNR of the proposed companding quantizer based on the second-degree spline approximation, ($SQNR^{SD}$) and the values of SQNR of quantizer based on the optimal compression function $c(x)$, ($SQNR^{OC}$), for a different number of levels *N*.

Table 3

The values of SQNR for the quantizer described in [4], the proposed companding quantizers and the optimal compandor

N	**$SQNR^{[4]}$ [dB]**	**$SQNR^{FD}$ [dB]**	**$SQNR^{SD}$ [dB]**	**$SQNR^{OC}$ [dB]**
16	19.36	19.51	19.69	20.22
32	25.33	25.35	25.80	26.01
64	31.08	31.07	31.88	31.89
128	36.82	36.74	37.80	37.81

Analyzing the results shown in Table 3 it can be noticed that the SQNR value of the proposed companding quantizer based on the first-degree spline approximation is very close to the SQNR value of the quantizer described in [4]. In [4] a piecewise uniform scalar quantizer is described whose thresholds of segments are equidistant and the number of reproduction levels is determined by optimizing the granular distortion, for the input signal modeled by a Gaussian PDF. As the proposed companding quantizer based on the second-degree spline approximation achieves SQNR up to 1.06 dB higher than the proposed companding quantizer based on the first-degree spline approximation, it can be concluded that the application of the second-degree spline approximation, for the same number of segments and the number of levels, provides higher SQNR than the quantizer described in [4]. With the increase of *N*, the difference in signal quality becomes more significant.

One of conclusion, that can be also observed, is that with the increase of *N* the $SQNR^{SD}$ approaches to the $SQNR^{OC}$, which isn't the case with the $SQNR^{FD}$. Thus, for $N = 64$ and $N = 128$, the scalar compandor of a Gaussian source whose compressor function is the second-degree spline approximation of the optimal compressor function can achieve SQNR arbitrarily close to that of the optimal compandor. Although the compressor function of this quantizer consists of parabola parts, it is easily to find the inverse compressor function, which is the main difficulty with the optimal compression function of a Gaussian source [1]. Withal the number of segments is small. It can be summarized that the proposed model is a very effective solution because it represents a simple quantizer model that achieves a high quality signal.

5. SUMMARY AND CONCLUSION

This paper demonstrates that the proposed method of the optimal compressor function approximation considerably reduces the optimal compandor design complexity. The reduction of the complexity design of the proposed model is reflected in the fact that it is not necessary to solve systems of integral equations to determine inverse compressor function. Another advantage of the proposed model is that its achieved signal quality is very close to the signal quality achieved with the optimal compandor. From the obtained results and on the bases of comparison to other models, it can be also concluded that for $N \geq 32$ the proposed model based on the second-degree spline functions achieves signal quality within of 0.2 dB in respect to that of optimal compander.

ACKNOWLEDGMENTS

This work is supported by Serbian Ministry of Education and Science through Mathematical Institute of Serbian Academy of Sciences and Arts (Project III44006) and by Serbian Ministry of Education and Science (Project TR32035).

Received on May 20, 2014

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