CONSIDERATIONS ON THE CURVATURE TENSOR IN THE GENERAL THEORY OF RELATIVITY

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Key words: Curvature tensor, Field equations, General theory of relativity.

In the General Theory of Relativity the curvature tensor has an important role in expressing the field equations. For this purpose a certain tensor is widely used. The expression of this tensor has been derived by requiring fulfilling of three conditions. The final condition to be satisfied requires the divergence of the tensor be identically zero. Finally an expression containing two terms was obtained. After a deeper analysis of the known formula, we can see that the divergence of each of both terms is identically zero, so that instead of a single expression, we could accept infinity of expressions. This circumstance appeared because the best deductions use general formulae leading to an expression (a sum of two terms) the divergence of which is zero. The aim of the paper is to establish general expressions avoiding the mentioned situation.

1. **INTRODUCTION**

In the General Theory of Relativity one considers arbitrary space-time transformations from a Galilean system of co-ordinates, K_{G} (provided with a Cartesian system of co-ordinates) to a general Gaussian system of co-ordinates, K (provided with any curvilinear system of co-ordinates). The assumed invariant in this transformation is the following quadratic form called quadratic fundamental form [1–6]:

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j \quad , \tag{1}$$

where the quantities g_{ij} are called the components of the fundamental covariant tensor:

$$g_{ij} = \sum_{u,v=0}^{N} \frac{\partial y^u}{\partial x^i}\frac{\partial y^v}{\partial x^j}, \tag{2}$$

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Rev. Roum. Sci. Techn. – Électrotechn. et Énerg., **53**, *4,* p. 359–365, Bucarest, 2009

where the variables of type y^{μ} denote the co-ordinates of the system K_G, whereas the variables of type x^i denote those of the system K. As known, in a Galilean system of reference, we have:

$$g_{00} = 1; \quad g_{ii} = -1; \quad i \in [1, 3]; \quad g_{ij} = 0, \forall i \neq j; \tag{3 a,b,c}$$

and then the corresponding quadratic form (1) is called normal or quasi-Euclidean form of the metric of the space defined by the co-ordinates x^i, or in order to avoid any confusion, defined by y^i, like above.

If the quadratic form is in any case different from that of formulae (3), it is called Riemannian form of the metric of the space defined by the co-ordinates x^i. If the fundamental quadratic form of any space cannot be reduced to the form having the components of formulae (3), that space is considered, by analogy with the theory of surfaces, as having a curvature [7, 8].

In real applications, the quantities g_{ij}, are not given directly by the considered system of bodies and the used system of reference, but result by solving certain field equations, so that, these quantities represent, to within a constant factor, potentials of forces.

2. **POSITION OF THE PROBLEM**

One of the aims of the General Theory of Relativity is to establish general laws of the gravitation field. As known, the equations of the gravitational field in the Newton theory are based, like the theory of Electrostatics, on the Poisson equation:

$$\operatorname{div}(\nabla\varphi) = \nabla^2\varphi = 4\,\pi\, k\rho\,, \tag{4}$$

where the following symbols have been used: φ – the gravitational potential, ρ – the specific mass, and k a certain constant.

In order to obtain a more general system of equations, Einstein proposed, [1], to replace the scalar potential equation (4) by a tensor equation containing instead of the derivatives of φ, the derivatives of a tensor formed of quantities g_{ij},. This tensor, according to this proposal, should satisfy the following conditions [1, p. 99, after formula (95)]:

1. It should not contain derivatives of an order greater than two.
2. It should be linear and homogeneous relatively to the derivatives of the second order.
3. Its divergence should be identically zero.

There is also added that it can be proved that the first two conditions are satisfied by the following expression:

$$R_{ij} + a g_{ij} R\,, \tag{5}$$

where R_{ij} is the Riemann tensor**;** R is given by the relation:

$$R = g^{ij} R_{ij} = R_i^i\,, \tag{6}$$

and a is a constant. Also, there is added that for the third condition to be satisfied, the constant a should be equal to $\frac{-1}{2}$. The same expression is given in the works of Fock [3, p. 248], Pathria [4, p. 201] and Ludvigsen [5, p. 99].

Let us examine for which values the mentioned condition will be satisfied. According to mentioned works, we shall assume that the following relation exists:

$$\mathrm{Div}\left(R_{ij} - \frac{1}{2} g_{ij} R \right) = \nabla_i \left(R_{ij} - \frac{1}{2} g_{ij} R \right) = 0\,. \tag{7}$$

Multiplying by g^{ij}, and taking into account relations (A.2 b) and (A.4 c), there follows:

$$\nabla_i \left[g^{ij} \left(R_{ij} - g_{ij} \frac{1}{2} R \right) \right] = \nabla_i \left(g^{ij} R_{ij} - \frac{1}{2} g^{ij} g_{ij} R \right) = \nabla_i \left(R - \frac{4}{2} R \right) = -\nabla_i R = 0\,. \tag{8}$$

From relations (7, 8), and (A.4 c), we shall obtain:

$$\nabla_i R = 0; \quad \nabla_i R_{ij} = 0\,. \tag{9}$$

There follows that the third condition will be satisfied regardless the value of the coefficient a. The expression used in literature:

$$G_{ij} = R_{ij} - g_{ij} \frac{1}{2} R\,, \tag{10}$$

has its divergence identical zero but each term of the expression satisfies the same condition. The reason is that the deduction presented in literature starts from the relation of Bianchi, in which the components appear together.

3. **CURVATURE TENSOR OF THE SPACE-TIME CONTINUUM**

We shall present a deduction of the curvature tensor. Let A^i be any contravariant vector, each component of which is continuous and its derivatives of the first two orders exist. Also, let us assume that each element of the fundamental tensor is continuous and its derivatives of the first three orders exist.

3.1. THE EXPRESSION OF THE CURVATURE TENSOR

Let us consider the covariant derivatives:

$$\nabla_r A^i = \frac{\partial A^i}{\partial x^r} + \Gamma^i_{ur} A^u ;, \tag{11}$$

where the left-hand side of each relation is a tensor of rank 2, once covariant and once contravariant. The following covariant derivatives will be expressed by using relation (A.6):

$$\begin{aligned}\nabla_s\left(\nabla_r A^i\right) = \frac{\partial^2 A^i}{\partial x^s \partial x^r} + \frac{\partial \Gamma^i_{ur}}{\partial x^s} A^u + \Gamma^i_{ur} \frac{\partial A^u}{\partial x^s} + \Gamma^i_{qs}\left(\frac{\partial A^q}{\partial x^r} + \Gamma^q_{ur} A^u\right) - \\ - \Gamma^q_{rs}\left(\frac{\partial A^i}{\partial x^q} + \Gamma^i_{uq} A^u\right),\end{aligned} \tag{12 a}$$

and

$$\begin{aligned}\nabla_r\left(\nabla_s A^i\right) = \frac{\partial^2 A^i}{\partial x^r \partial x^s} + \frac{\partial \Gamma^i_{us}}{\partial x^r} A^u + \Gamma^i_{us} \frac{\partial A^u}{\partial x^r} + \Gamma^i_{qr}\left(\frac{\partial A^q}{\partial x^s} + \Gamma^q_{us} A^u\right) - \\ - \Gamma^q_{sr}\left(\frac{\partial A^i}{\partial x^q} + \Gamma^i_{uq} A^u\right).\end{aligned} \tag{12 b}$$

Now, in order to obtain a simple expression we shall subtract the last two equations side by side, and we shall take into consideration that in the terms in which certain indices occur twice, but in different order, they may be replaced by each other. There follows:

$$\left(\nabla_s \nabla_r - \nabla_r \nabla_s\right) A^i = \left(\frac{\partial \Gamma^i_{ur}}{\partial x^s} - \frac{\partial \Gamma^i_{us}}{\partial x^r}\right) A^u + \left(\Gamma^i_{qs}\Gamma^q_{ur} - \Gamma^i_{qr}\Gamma^q_{us}\right) A^u . \tag{13}$$

The expression (13) may be written:

$$\left(\nabla_s \nabla_r - \nabla_r \nabla_s\right) A^i = R^i_{rsu} A^u . \tag{14}$$

The right-hand side of expression (14), apart from the contravariant vector as factor, satisfies the first two conditions, and is just the Riemann curvature tensor, in which we have separated, by parentheses, two terms:

$$R^i_{rsu} = \left(\frac{\partial \Gamma^i_{ur}}{\partial x^s} - \frac{\partial \Gamma^i_{us}}{\partial x^r}\right) + \left(\Gamma^i_{qs}\Gamma^q_{ur} - \Gamma^i_{qr}\Gamma^q_{us}\right). \tag{15}$$

By contracting with respect to indices i and r, we shall obtain:

$$R_{su} = \left(\frac{\partial \Gamma^r_{ur}}{\partial x^s} - \frac{\partial \Gamma^r_{us}}{\partial x^r} \right) + \left(\Gamma^r_{qs} \Gamma^q_{ur} - \Gamma^r_{qr} \Gamma^q_{us} \right). \tag{16}$$

3.2. THE DIVERGENCE OF THE CURVATURE TENSOR

The proofs, aiming the calculation of the divergence of the curvature tensor are based on Bianchi identity and that of the geodesic co-ordinates, like in works [1-5].

We shall give a proof starting from the same identity and co-ordinates, but the analysis carried out will lead to different result and conclusion.

We should remark that expression (16) could not be used directly for the calculation of its covariant derivative, since apart from the indices s and u, which could be considered as fixed, it also contains other indices corresponding to all co-ordinates. Therefore, we shall go back and consider a relation of type (15).

As known, in geodesic co-ordinates the Christoffel symbols are zero, but their derivatives with respect to co-ordinates may be different from zero. Therefore the covariant derivative of expression (15) will concern only the first parenthesis and will be:

$$\nabla_k R^i_{rsu} = \frac{\partial^2 \Gamma^i_{ur}}{\partial x^k \partial x^s} - \frac{\partial^2 \Gamma^i_{us}}{\partial x^k \partial x^r} . \tag{17}$$

Using relation (17), there follows:

$$\nabla_k R^i_{rsu} + \nabla_s R^i_{ruk} + \nabla_u R^i_{rks} = 0 , \tag{18}$$

which is the Bianchi identity.

According to relation (17), the identity (18) may be written:

$$-\nabla_k R^i_{sru} + \nabla_s R^i_{ruk} + \nabla_u R^i_{rks} = 0 . \tag{18 a}$$

By putting $i = u$, and contracting with respect to index i, there follows:

$$-\nabla_k R_{sr} + \nabla_s R_{rk} + \nabla_i R^i_{rks} = 0 . \tag{19}$$

Multiplying the last relation by g^{rs}, and taking into account relation (6), we shall obtain:

$$-\nabla_k R^s_s + \nabla_s R^s_k + \nabla_i R^i_k = 0 \tag{20}$$

and replacing index s by i in the second term of the left-hand side of relation (20), both taking all values from zero to 3, one obtains:

$$-\nabla_k R_s^s + 2\nabla_i R_k^i = 0 \tag{21}$$

or taking into account relations (A.1 a, b):

$$-\nabla_i g_k^i R_s^s + 2\nabla_i R_k^i = 0; \quad \nabla_i\left(-g_k^i R + 2R_k^i\right) = 0. \tag{22 a, b}$$

Multiplying relation (22 b) by g_i^k, and taking into account relations (6) and (A.3 a, b), we shall obtain:

$$\nabla_i\left(-4R + 2R_i^i\right) = 0; \quad \nabla_i\left(-4R + 2R\right) = 0; \quad \nabla_i R = 0. \tag{23 a, b, c}$$

Multiplying relation (22 b) by g_{is}, and considering relations (A.2 a), (A.4 b) and (23 c), we shall obtain:

$$\nabla_i R = 0; \quad \nabla_i R_{ks} = 0. \tag{24 a, b}$$

The last relation is different from those known in literature and confirms our remark of Section 2 of the present paper.

APPENDIX

According to formulae of Appendices A1.1, A.1.2 and A.2 of [6], and starting from those formulae, we shall obtain:

$$g^{uj} g_{ui} = g_i^j; \quad g_i^j = 1, \quad \forall i = j; \quad g_i^j = 0, \quad \forall i \neq j. \tag{A.1 a, b}$$

$$g^{ij} g_{ij} = g_i^i, \quad i = \text{const}; \quad g^{ij} g_{ij} = 4, \quad \forall i, j \in [0, 3]. \tag{A.2 a, b}$$

$$g_i^j g_j^i = 1, \quad i = \text{const}; \quad g_i^j g_j^i = 4, \quad \forall i, j \in [0, 3]. \tag{A.3 a, b}$$

$$\frac{\partial g_{ij}}{\partial x_k} = \Gamma_{i,jk} + \Gamma_{j,ik}; \quad \nabla_k g_{ij} = 0; \quad \nabla_k g^{ij} = 0, \quad \forall i, j, k \in [0, 3]. \tag{A.4 a, b, c}$$

$$\nabla_k g_i^j = 0, \quad \forall i, j \in [0, 3]. \tag{A.5}$$

$$\nabla_k C_j^i = \frac{\partial C_j^i}{\partial x_k} + \Gamma_{lk}^i C_j^l - \Gamma_{jk}^l C_l^i. \tag{A.6}$$

All formulae of this Appendix are useful for the transformations occurring in the previous treatment.

Received on June 27, 2008

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