A NEW APPROACH TO SYNCHRONIZE CHAOTIC MAPS WITH DIFFERENT DIMENSIONS

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In recent years the synchronization of a chaotic response system with a chaotic drive system has attracted great interest in nonlinear science and engineering, due to the potential applications in chaos-based communications. This paper focuses on a new approach to achieve synchronization between two chaotic (hyperchaotic) maps with different dimensions. In particular, given a drive system map with dimension \( n \) and a response system with dimension \( m \), the proposed approach enables each response system state to be synchronized with any linear combination of the drive system states. The method is based on a theorem that assures synchronization is achieved under certain broad conditions involving nilpotent matrices and suitable coupling between the two maps. The approach presents some new useful features, including the fact that exact synchronization up to an arbitrary scaling matrix is achievable in finite time for a wide class of chaotic maps with different dimensions. Moreover, the method is rigorous, systematic and readily implemented. The technique is illustrated by examples of synchronization between the three-dimensional case of the generalized Henon map and a recently introduced two-dimensional quadratic map. These examples highlight that exact synchronization is effectively achieved in finite time for any scaling matrix.

1. INTRODUCTION

Synchronizing a chaotic response system with a chaotic drive system has attracted great interest in nonlinear science and engineering, by virtue of the potential applications in chaos-based communications [1–10]. Most of the synchronization methods inspired by Carroll and Pecora [1] focus on complete (identical) synchronization, where two identical chaotic systems asymptotically approach the same trajectory [2–7]. Other types of synchronization have been proposed in the literature [8–17]. Among these, projective synchronization [11] provides response system variables are scaled replicas of the drive system variables [12]. Since the scaling factor is not predictable in [11], control methods have been successively developed for choosing any desired scaling factor [12–17]. A variation of projective synchronization is the so-called full state hybrid projective synchronization (FSHPS) [18–22]. In this type of synchronization the scaling factor can be different.
for each state variable, meaning that the single scaling parameter originally introduced in [11] is replaced by a diagonal scaling matrix [18–23]. In particular, in [22] active control is applied to synchronize a class of chaotic maps, including the so-called Grassi-Miller map, whereas in [23] the Lyapunov stability theory is utilized to achieve FSHPS (called therein modified generalized projective synchronization) for a new chaotic system.

Recently, an interesting scheme [24, 25] has been introduced, in which each drive system state achieves synchronization with any arbitrary linear combination of response system states. Since the adoption of an arbitrary scaling matrix represents a generalization of FSHPS (where the scaling matrix is a diagonal one), the method in Ref. [24] has been called arbitrary full-state hybrid projective synchronization, indicating that drive and response systems achieve projective synchronization up to an arbitrary scaling matrix. The method in [24, 25] can be only applied to identical chaotic maps, i.e., the drive and response systems must be described by the same difference equations, meaning that the two maps are identical and, consequently, have the same dimension. Based on these considerations, this paper makes a contribution to the topic of synchronization by introducing a further generalization, where a n-dimensional chaotic map is synchronized with a different m-dimensional map up to an arbitrary scaling matrix. Specifically, the new approach developed herein enables synchronization between the two different maps to be exactly achieved in finite time, i.e., for a m-dimensional driven map the error is exactly zero after m steps, for any arbitrary scaling matrix. Moreover, the method is straightforward and can be applied to a wide class of chaotic discrete-time systems. The paper is organized as follows. At first a theorem is proved, which assures that exact synchronization is achieved in finite time under certain broad conditions involving nilpotent matrices and suitable coupling between the two maps. The capability of the method is illustrated by examples of synchronization (up to an arbitrary scaling matrix) between the three-dimensional Grassi-Miller map [22] and the two-dimensional quadratic map recently proposed in [26].

2. SYNCHRONIZING MAPS WITH DIFFERENT DIMENSIONS UP TO AN ARBITRARY SCALING MATRIX

The drive systems considered here are n-dimensional maps described by the following state equation:

$$x(k+1) = Ax(k) + bf(x(k)), \quad (1)$$

where $x(k) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times 1}$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

On the other hand, the response systems considered here are m-dimensional maps described by the state equation:
where $w(k+1) = Cw(k) + dg(w(k)) + u(w(k), x(k)),$ \( (2) \)

where $w(k) \in \mathbb{R}^m,$ $C \in \mathbb{R}^{m \times m},$ $d \in \mathbb{R}^m,$ $g : \mathbb{R}^m \to \mathbb{R},$ and $u(w, x) \in \mathbb{R}^m$ is the coupling to be determined in order to achieve synchronized dynamics.

Note that the map (1) and the uncoupled map (2) (i.e., the map with $u = 0$ in (2)) are characterized by completely different dynamics, particularly given that they have different dimensions $n$ and $m$, respectively. By extending the definition reported in [24], the maps described by (1) and (2) are said to be synchronized up to an arbitrary scaling matrix when there exists a rectangular $m \times n$ matrix

\[
\alpha = \begin{bmatrix}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\
\vdots & \ddots & \cdots & \vdots \\
\alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn}
\end{bmatrix} \in \mathbb{R}^{m \times n} \tag{3}
\]

such that

\[
\|e(k)\| = \|w(k) - \alpha x(k)\| \to 0 \text{ as } k \to \infty. \tag{4}
\]

The condition (4) means that each response system variable $w_i(k)$ synchronizes with a linear combination of drive system variables $\alpha_{i1}x_1(k) + \alpha_{i2}x_2(k) + \cdots + \alpha_{in}x_n(k)$, for any $i = 1, 2, \ldots, m$. Note that the scaling matrix (3) generalizes the definition reported in [24] since the scaling matrix can be rectangular rather than square. As shown next, this precondition will lead to a novel scheme, where the maps to be synchronized may have different dimensions.

Before introducing the method, some definitions and properties related to the nilpotent matrices [27] are briefly summarized. Let $C$ and $d$ be the matrices defined in (2), and let $k \in \mathbb{R}^{l \times n}$ be a suitable gain vector. The square $mxm$ matrix $[C - dk]$ is said to be a nilpotent matrix if $[C - dk]^\rho = 0$ for some nonnegative integer $\rho \leq m$ [27]; the smallest $\rho$ for which this is true is the degree of the nilpotent matrix [27].

A remarkable property of a nilpotent matrix is that all its eigenvalues are zero [27]. Now, a theorem that synchronization up to an arbitrary scaling matrix is exactly achievable in finite time using a suitable coupling $u(w, x)$ can be developed.

**THEOREM.** Given the $n$-dimensional drive system (1) and the $m$-dimensional response system (2), if the matrix

\[
\begin{bmatrix}
d & Cd & C^2d & C^3d & \cdots & C^{m-1}d
\end{bmatrix}
\]

is full rank and if the control $u(w, x)$ is selected as

\[
u(w(k), x(k)) = -C(\alpha x(k)) - dk w(k) + dk(\alpha x(k)) - dg(w(k)) + \alpha x(k + 1) \tag{6}
\]
then exact synchronization up to an arbitrary scaling matrix is achieved in finite time between systems (1) and (2), provided that the gain vector $\mathbf{k} \in \mathbb{R}^{1 \times n}$ makes the matrix $[\mathbf{C} - d\mathbf{k}]$ nilpotent.

The proof can be readily obtained by deriving the error system and by taking into account the results on eigenvalue placement reported in [9]. This method is widely applicable since the matrix (5) is full rank for a wide class of chaotic (hyperchaotic) maps in the form (2) with $u = 0$. In particular, the class characterized by full rank matrix in the form (5) includes a number of maps reported in [28], such as the logistic map, the cubic map, the Duffing map, the Gauss map and the Lozi map. Moreover, the generalized Henon map [29], its three dimensional case referred to as the Grassi-Miller map [22], the gingerbreadman map [30], the double scroll map [31] and the minimal quadratic map illustrated in [26] all are characterized by a full rank matrix in the form (5). As a consequence, according to previous theorem, all these maps can achieve exact synchronization up to an arbitrary scaling in finite time. The proposed scheme is characterized by the following useful and remarkable features:

- It is rigorous and systematic: the first feature results from the fact that the method is based on a theorem, while most techniques rely on computation of Lyapunov exponents to verify synchronization. The second feature results from the fact that, by using the coupling (6) and by making the matrix $[\mathbf{C} - d\mathbf{k}]$ nilpotent, it is possible to synchronize in a systematic way (i.e., based on these specific steps) all the maps belonging to the class described by (1) and (2), respectively.

- It enables exact synchronization to be achieved in finite time. This property follows from dead-beat control theory [9], assuring that the error dynamics will reach exactly zero in at most $m$ steps. Note that some previous FSHPS methods, such as those described in [19, 22], do not provide this feature, since the error asymptotically tends toward zero when $k$ goes to infinity.

- It does not require computation of Lyapunov exponents to validate synchronization, since the technique guarantees exact synchronization. Namely, in contrast to most of the techniques developed so far that rely on numerical methods to verify synchronization, the proposed approach is based on a theorem, which assures exact synchronization when the matrix $[\mathbf{C} - d\mathbf{k}]$ is made nilpotent.

Remark 1. Even though the synchronization tool proposed herein may appear quite unachievable from the engineering point of view, it should be pointed out that an approach to achieve exact synchronization of hyperchaotic maps in finite time has been already developed in hardware by the author of the present manuscript (see [9]). Specifically, in reference [9] an electronic implementation (using sample and hold circuits as well as standard operational amplifier configurations) has been developed. The results in [9] clearly prove that exact synchronization in finite time can be experimentally achieved. Even though the synchronization approach in [9]
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does not present scaling capabilities or maps with different dimensions (as the one proposed herein), nevertheless it demonstrates the feasibility of engineering implementations of exact synchronization schemes.

Remark 2. Even though the main focus in the present manuscript is the development of a theoretical approach for synchronizing chaotic maps with different dimensions, nevertheless it should be pointed out that exact synchronization in finite time can be exploited for secure communications purposes [30]. Namely, in reference [30] it is shown that, given a cascade connection of chaotic maps, exact synchronization in finite time is achievable between pairs of drive–response systems. This “propagated synchronization” starts from the innermost drive–response system pair and propagates toward the outermost drive-system pair. Choosing the drive-system input to be an information signal (encrypted via an arbitrary encryption function) yields a potential application of the architecture in chaos-based secure communications [30].

3. SYNCHRONIZATION EXAMPLES

Some examples of synchronization up to an arbitrary scaling are now illustrated. The considered drive system is the three-dimensional case of the generalized Henon map [29] referred to as the hyperchaotic Grassi-Miller map [22], which was implemented as an electronic circuit by the author of the present manuscript in [9]. The dynamics of this map can be written in the form (1) as:

\[
\begin{bmatrix}
    x_1(k+1) \\
    x_2(k+1) \\
    x_3(k+1)
\end{bmatrix} =
\begin{bmatrix}
    0 & 0 & -0.1 \\
    1 & 0 & 0 \\
    0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
    x_1(k) \\
    x_2(k) \\
    x_3(k)
\end{bmatrix} +
\begin{bmatrix}
    1 \\
    0 \\
    0
\end{bmatrix}
\left(1.76 - (x_2(k))^2\right).
\] (7)

The map exhibits the hyperchaotic attractor shown in Fig. 1. The response system is selected as the two-dimensional quadratic map, with a quasi-periodic route to chaos, recently introduced in [26]. The map can be written in the form (2) as:

\[
\begin{bmatrix}
    w_1(k+1) \\
    w_2(k+1)
\end{bmatrix} =
\begin{bmatrix}
    b & 0 \\
    1 & 0
\end{bmatrix}
\begin{bmatrix}
    w_1(k) \\
    w_2(k)
\end{bmatrix} +
\begin{bmatrix}
    1 \\
    0
\end{bmatrix}
\left(1 - a(w_2(k))^2\right) +
\begin{bmatrix}
    u_1(w_1(k), w_2(k), x_1(k), x_2(k), x_3(k)) \\
    u_2(w_1(k), w_2(k), x_1(k), x_2(k), x_3(k))
\end{bmatrix}
\] (8)

where \(a\) and \(b\) are bifurcation parameters.

When \(a = 0.7\) and \(b = 0.9\), the uncontrolled map (8) (i.e., the map with \(u_1 = u_2 = 0\)) exhibits the chaotic attractor reported in Fig. 2 (left).
By choosing the scaling matrix (3) as:

\[
\begin{bmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23}
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}
\] (9)

the coupling (6) can be written as:

\[
\begin{bmatrix}
u_1(k) \\
u_2(k)
\end{bmatrix} = \begin{bmatrix} 0.9 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\
x_2(k) \\
x_3(k)
\end{bmatrix}^T - \begin{bmatrix} 1 \\
0 
\end{bmatrix} k_1 \begin{bmatrix} w_1(k) \\
w_2(k) 
\end{bmatrix} + \begin{bmatrix} 1 \\
0 
\end{bmatrix} k_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\
x_2(k) \\
x_3(k)
\end{bmatrix}^T - \begin{bmatrix} 1 \\
0 
\end{bmatrix} \left(1 - 0.7(w_2(k))^2\right) + \begin{bmatrix} 1 \\
0 
\end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k+1) \\
x_2(k+1) \\
x_3(k+1)
\end{bmatrix}^T,
\] (10)

where \(k_1, k_2\) is the gain vector to be determined. By considering the drive system (7), the response system (8), the scaling matrix (9) and the coupling (10), it is straightforward to show that the error dynamics are described by

\[
\begin{bmatrix}
e_1(k+1) \\
e_2(k+1)
\end{bmatrix} = \begin{bmatrix} 0.9 & 0 \\ 1 & 0 
\end{bmatrix} \begin{bmatrix} k_1 \\
k_2 
\end{bmatrix} \begin{bmatrix} e_1(k) \\
e_2(k)
\end{bmatrix}.
\] (11)

Since the matrix (5) is full rank, it can be readily shown that the gain vector \(k = [0.9, 0]\) makes the matrix

\[
\begin{bmatrix} 0.9 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 
\end{bmatrix}
\] (12)
nilpotent of degree 2, given that
\[
\begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}^2 = 0.
\] (13)

By virtue of previous theorem, exact synchronization up to an arbitrary scaling is achieved in finite time between systems (7) and (8), that is, the error dynamics will reach exactly zero after two steps. In order to show the effectiveness of the approach, Fig. 2 (right) shows the attractor of the response system (8) in the \((w_1, w_2)\)-plane when the coupling is given by (10) with \(k = [0.9 \ 0]\).

Referring to Fig. 2, note that the effect of the proposed synchronization scheme is to produce a two-dimensional response system attractor (right) that is completely different from both the uncontrolled response system attractor (left) and the three-dimensional drive system attractor (Fig. 1). The error (4) between systems (7) and (8) is reported in Table 1. This table confirms that the error is exactly zero from step three on, indicating that exact synchronization up to an arbitrary scaling is achieved in finite time between systems (7) and (8).

**Table 1**
The errors as a function of \(k\) (scaling matrix in (9)).
The errors are exactly zero from step three on

<table>
<thead>
<tr>
<th>(k)</th>
<th>(e_1(k) = w_1(k) - x_1(k))</th>
<th>(e_2(k) = w_2(k) - (x_2(k) + x_3(k)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>step1</td>
<td>1.156</td>
<td>0.209</td>
</tr>
<tr>
<td>step2</td>
<td>0</td>
<td>1.194</td>
</tr>
<tr>
<td>step3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>step4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>step5</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
An additional example of synchronization is now given, using a different chaotic attractor generated by the map (8). Namely, when \( a = 1 \) and \( b = 0.3 \), the map (8) with \( u_1 = u_2 = 0 \) exhibits the chaotic attractor reported in Fig. 3 (left). Since the matrix (5), obtained from (8) when \( a = 1 \) and \( b = 0.3 \), is full rank, it can be readily shown that the gain vector \( k = \begin{bmatrix} 0.3 & 0 \end{bmatrix} \) makes the matrix

\[
\begin{bmatrix}
0.3 & 0 \\
1 & 0
\end{bmatrix} - \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix} 0.3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\]

(14)

nilpotent of degree 2 (see also (13)). By taking the following scaling matrix

\[
\begin{bmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23}
\end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix},
\]

(15)

the control (6) can be written as:

\[
\begin{bmatrix}
u_1(k) \\
u_2(k)
\end{bmatrix} = \begin{bmatrix} 0.3 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(k) \\ w_2(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.3 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left(1 - \frac{w_1(k)}{\alpha_{11}}\right)^2 \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix}.
\]

(16)

The chaotic attractor of the response system (8) with the control (16) is shown in Fig. 3 (right). Notice that the effect of synchronizing (up to a scaling matrix) maps of different dimensions is to produce a two-dimensional response system attractor (Fig. 3, right) that is completely different from both the uncontrolled response system attractor (Fig. 3, left) and the three-dimensional drive system attractor (Fig. 1).

Fig. 3 – Chaotic attractor of the map (8) in the \((w_1, w_2)\)-plane for \( a = 1 \), \( b = 0.3 \) and \( u = 0 \) (left); chaotic attractor of the map (8) in the \((w_1, w_2)\)-plane when the coupling is given by (16) (right).
Notice that the condition to be checked in order to verify synchronization is straightforward (i.e., the matrix $[C - dk]$ must be nilpotent via a suitable $k$), while other synchronization approaches in literature usually require the Lyapunov exponents to be computed [32–34].

4. CONCLUSIONS

This paper has illustrated a new scheme to achieve synchronization between two chaotic maps with different dimensions for any arbitrary scaling matrix. A developed theorem assures that synchronization is achievable under certain broad conditions involving nilpotent matrices and suitable coupling between the two maps. The approach is characterized by several new useful properties, including the fact that exact synchronization is achievable in finite time for a wide class of chaotic maps with different dimensions. To the best of the author’s knowledge, no other method in literature presents the same features. The capability of the technique has been illustrated by synchronizing the three-dimensional Grassi-Miller map and a recently introduced two-dimensional quadratic map.

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