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***Dedicated to the memory of Academician Andrei Ţugulea***

# NEW ANALYTIC SOLUTION FOR THE MAGNETIZATION OF TWO SPHERES

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**The boundary value problem of two ferromagnetic spheres introduced in a uniform magnetic field is solved by using scalar potentials and bispherical coordinates. Under the approximation of an ideal ferromagnetic material, the potential of the outside field is constant over the sphere surfaces and the resultant field has the same structure as the electrostatic field of two uncharged conducting spheres in a uniform electric field. The magnetic flux density inside each sphere is derived from a Laplacian scalar potential by imposing the continuity of the normal component of flux density through the sphere surfaces. The constants of integration of this potential are determined from an infinite tridiagonal matrix equation which yields a computationally efficient solution, of controllable accuracy.**

## 1. INTRODUCTION

A complete analytical treatment of the classical problem of two conducting spheres in a uniform electrostatic field, including numerical results of a controllable accuracy for the field intensity and for the force between the spheres, was presented in [1]. The sphere surfaces are taken to be coordinate surfaces in a bispherical coordinate system. The problem of two superconducting spheres in a uniform magnetic field was solved in [2], the coefficients in the solution series expansions being determined from infinite band matrix equations. A field analysis for systems of ferromagnetic spheres in the presence of uniform magnetic fields has been performed in [3] by considering the material of the spheres to be ideal ferromagnetic (*i.e.*, linear, with a permeability $\to \infty$) and by using Laplacian scalar potentials in spherical coordinates both outside and inside the spheres. In the field outside the spheres the potential is constant over the surface of each sphere and is obtained by applying appropriate translational addition theorems [4]. The coefficients in the solution series satisfy infinite systems of algebraic equations with dense matrices.

In the present paper, the problem of two ideal ferromagnetic spheres in a uniform magnetic field is treated as a boundary value problem by employing bispherical coordinates. The potential outside the spheres is expressed in terms of single series, as in [1] for the case of two uncharged spherical conductors in a uniform electrostatic field, while the coefficients in the series expansion of the scalar potential inside each sphere are determined by only solving a tridiagonal matrix equation. To the best of our knowledge such a solution is not available in the literature. The solution presented is useful for an efficient treatment of the motion of magnetic particles subjected to both magnetic and hydrodynamic forces in ferrohydrodynamics applications [5]. On the other hand, the analysis of the fields inside the two ferromagnetic spheres yields efficient benchmark expressions which can be used for the evaluation of their magnetization and of internal forces and stresses.

## 2. MAGNETIC SCALAR POTENTIAL FORMULATION

Consider two ferromagnetic spheres, as shown in Fig. 1, whose common axis is chosen to be the *z*–axis. The relationship between the bispherical coordinates $\alpha, \beta, \varphi$ and the Cartesian coordinates *x*, *y*, *z* is [6]

$$x = \frac{c\sin\alpha\cos\varphi}{\cosh\beta - \cos\alpha},\ y = \frac{c\sin\alpha\sin\varphi}{\cosh\beta - \cos\alpha},\quad z = \frac{c\sinh\beta}{\cosh\beta - \cos\alpha}. \tag{1}$$

The sphere surfaces have the equations $\beta = -\beta_1$ and $\beta = \beta_2$ with

$$\sinh\beta_1 = \frac{c}{a_1},\ \sinh\beta_2 = \frac{c}{a_2},\quad \sqrt{c^2 + a_1^2} + \sqrt{c^2 + a_2^2} = d, \tag{2}$$

where $c$ is the semi-focal distance and $d$ the distance between the centers of the two spheres. Media outside and inside the spheres are linear, isotropic and homogeneous, the medium outside being nonmagnetic. The spheres are in the presence of a uniform magnetic field whose flux density $\boldsymbol{B}_0$ is oriented along the *z*–axis. A scalar potential is used in what follows both outside and inside the spheres, which is defined such that its negative gradient is everywhere equal to the magnetic flux density.

### 2.1. POTENTIAL OUTSIDE THE SPHERES

For $-\beta_1 \le \beta \le \beta_2$ the Laplacian scalar potential is expressed in the form [6]

$$\Psi(\alpha,\beta) = (\cosh\beta - \cos\alpha)^{1/2} \times \sum_{n=0}^{\infty}\left[C_n^{(1)} \mathrm{e}^{-\left(n+\frac{1}{2}\right)\beta} + C_n^{(2)} \mathrm{e}^{\left(n+\frac{1}{2}\right)\beta}\right] P_n(\cos\alpha) - B_0 z, \tag{3}$$

where

$$|z| = \sqrt{2}c(\cosh\beta - \cos\alpha)^{1/2} \times \sum_{n=0}^{\infty}(2n+1)\mathrm{e}^{-\left(n+\frac{1}{2}\right)|\beta|} P_n(\cos\alpha) \tag{4}$$

and $P_n$ are Legendre polynomials. The constants of integration $C_n^{(1),(2)}$ are obtained from the condition that the surfaces of the two spheres are equipotential, *i.e.*,

$$\Psi(\alpha,-\beta_1)=\Psi_1,\quad \Psi(\alpha,\beta_2)=\Psi_2\,, \tag{5}$$

$\Psi_1$ and $\Psi_2$ being the unknown values of the potentials of the two sphere surfaces. Using the expression

$$(\cosh\beta-\cos\alpha)^{-1/2}=\sqrt{2}\sum_{n=0}^{\infty}(2n+1)e^{-\left(n+\frac{1}{2}\right)|\beta|}P_n(\cos\alpha) \tag{6}$$

one obtains

$$C_n^{(1)}=\frac{\sqrt{2}}{e^{(2n+1)(\beta_1+\beta_2)}-1}\Big[-B_0c(2n+1)\left(e^{(2n+1)\beta_2}+1\right) +\Psi_1 e^{(2n+1)\beta_2}-\Psi_2\Big], \tag{7}$$

$$C_n^{(2)}=\frac{\sqrt{2}}{e^{(2n+1)(\beta_1+\beta_2)}-1}\Big[B_0c(2n+1)\left(e^{(2n+1)\beta_1}+1\right) +\Psi_2 e^{(2n+1)\beta_1}-\Psi_1\Big]. \tag{8}$$

The values of the potentials $\Psi_1$ and $\Psi_2$ are determined by imposing the condition of zero total magnetic flux through each sphere, *i.e.*, the total true magnetic charge of each sphere is equal to zero,

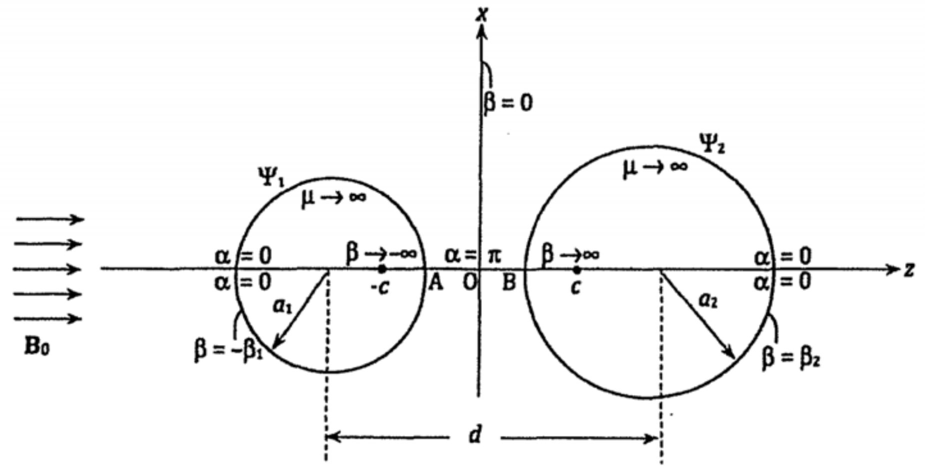


Fig. 1 – Two magnetic spheres in a uniform magnetic field.

$$\int_{\alpha=0}^{\pi}\int_{\varphi=0}^{2\pi}\left(\frac{\partial\Psi}{\partial n}\mathrm{d}S\right)_{\beta=-\beta_1,\beta_2}=0\,. \tag{9}$$

Using the scaling

$$\frac{\partial}{\partial n}=-\frac{1}{c}(\cosh\beta-\cos\alpha)\frac{\partial}{\partial\beta}\,. \tag{10}$$

the derivatives along the external normal at the sphere surfaces are derived in the form

$$\left(\frac{\partial\Psi}{\partial n}\right)_{\beta=-\beta_1,\beta_2}=-\frac{1}{c}(\cosh\beta_{1,2}-\cos\alpha)^{3/2} \times\sum_{n=0}^{\infty}(2n+1)e^{\left(n+\frac{1}{2}\right)\beta_{1,2}}C_n^{(1),(2)}P_n(\cos\alpha)\,. \tag{11}$$

With

$$(\mathrm{d}S)_{\beta=-\beta_1,\beta_2}=\frac{c^2\sin\alpha}{(\cosh\beta_{1,2}-\cos\alpha)^2}\mathrm{d}\alpha\mathrm{d}\varphi\,, \tag{12}$$

the two equations in (9) become

$$\int_{\alpha=0}^{\pi}\frac{\sin\alpha}{(\cosh\beta_{1,2}-\cos\alpha)^{1/2}}\sum_{n=0}^{\infty}(2n+1)e^{(2n+1)\beta_{1,2}} \times C_n^{(1),(2)}P_n(\cos\alpha)\mathrm{d}\alpha=0\,. \tag{13}$$

Finnaly, using again (6) and the orthogonality of the Legendre polynomials, one obtains

$$\sum_{n=0}^{\infty}C_n^{(1),(2)}=0\,, \tag{14}$$

which yields a system of two algebraic equations in $\Psi_1$ and $\Psi_2$ whose solution is

$$\Psi_1=B_0c\frac{[S_1(\beta_2)+S_1(0)]S_0(\beta_1)}{S_0(\beta_1)S_0(\beta_2)-(S_0(0))^2}-\frac{[S_1(\beta_1)+S_1(0)]S_0(0)}{S_0(\beta_1)S_0(\beta_2)-(S_0(0))^2}\,, \tag{15}$$

$$\Psi_2=-B_0c\frac{[S_1(\beta_1)+S_1(0)]S_0(\beta_2)}{S_0(\beta_1)S_0(\beta_2)-(S_0(0))^2}-\frac{[S_1(\beta_2)+S_1(0)]S_0(0)}{S_0(\beta_1)S_0(\beta_2)-(S_0(0))^2}\,, \tag{16}$$

where

$$S_0(\xi)=\sum_{n=0}^{\infty}\frac{e^{(2n+1)\xi}}{e^{(2n+1)(\beta_1+\beta_2)}-1}\,, \tag{17}$$

$$S_1(\xi)=S_0(\xi)+2\sum_{n=0}^{\infty}\frac{ne^{(2n+1)\xi}}{e^{(2n+1)(\beta_1+\beta_2)}-1}\,. \tag{18}$$

The series in (17) and (18) are rapidly convergent when evaluated as in [1].

## 2.2. POTENTIAL INSIDE THE SPHERES

*Inside the sphere* $\beta=-\beta_1$ the Laplacian magnetic scalar potential is expressed as

$$\Psi_i^{(1)}(\alpha,\beta)=(\cosh\beta-\cos\alpha)^{1/2}\times\sum_{n=0}^{\infty}L_n^{(1)}\mathrm{e}^{\left(n+\frac{1}{2}\right)\beta}P_n(\cos\alpha). \tag{19}$$

The constants of integration $L_n^{(1)}$ are determined by imposing the continuity of the normal components of flux density through $\beta=-\beta_1$, *i.e.*,

$$\left(\frac{\partial\Psi_i^{(1)}}{\partial n}\right)_{\beta=-\beta_1}=\left(\frac{\partial\Psi}{\partial n}\right)_{\beta=-\beta_1}. \tag{20}$$

From (19), with (10), we get

$$\left(\frac{\partial\Psi_i^{(1)}}{\partial n}\right)_{\beta=-\beta_1}=-\frac{1}{2c}(\cosh\beta_1-\cos\alpha)^{1/2}\times\sum_{n=0}^{\infty}L_n^{(1)}\left[\sinh\beta_1+(\cosh\beta_1-\cos\alpha)(2n+1)\right]\times\mathrm{e}^{-\left(n+\frac{1}{2}\right)\beta_1}P_n(\cos\alpha) \tag{21}$$

and, thus, the condition in (20) becomes (see (11))

$$\begin{aligned}&\sum_{n=0}^{\infty}L_n^{(1)}\mathrm{e}^{-\left(n+\frac{1}{2}\right)\beta_1}\left[\sinh\beta_1+(2n+1)\cosh\beta_1\right]P_n(\cos\alpha)\\&-\cos\alpha\sum_{n=0}^{\infty}L_n^{(1)}(2n+1)\,\mathrm{e}^{-\left(n+\frac{1}{2}\right)\beta_1}P_n(\cos\alpha)=\\&=2\cosh\beta_1\sum_{n=0}^{\infty}C_n^{(1)}(2n+1)\,\mathrm{e}^{\left(n+\frac{1}{2}\right)\beta_1}P_n(\cos\alpha)-\\&-2\cos\alpha\sum_{n=0}^{\infty}C_n^{(1)}(2n+1)\,\mathrm{e}^{\left(n+\frac{1}{2}\right)\beta_1}P_n(\cos\alpha).\end{aligned} \tag{22}$$

To have both sides arranged as standard series of Lengendre polynomials $P_n(\cos\alpha)$, we employ the recurrence relation [7]

$$\cos\alpha P_n(\cos\alpha)=\frac{1}{2n+1}\left[(n+1)P_{n+1}(\cos\alpha)+nP_{n-1}(\cos\alpha)\right], \tag{23}$$

which we write in the form

$$\cos\alpha P_n(\cos\alpha)==\frac{1}{2n+1}\sum_{n=0}^{\infty}\left[(n+1)\delta_{n,l-1}+n\delta_{n,l+1}\right]P_l(\cos\alpha), \tag{24}$$

where $\delta$ is the Kronecker symbol. Substituting (24) in (22) and identifying the coefficients of the Legendre polynomials series in the two sides yields, finally, the following infinite tridiagonal matrix equation for the coefficients $L_n^{(1)}$

$$\begin{bmatrix} b_0^{(1)} & c_1^{(1)} & & & \ddots & 0\\ a_0^{(1)} & b_1^{(1)} & c_2^{(1)} & & & \ddots\\ \ddots & a_1^{(1)} & b_2^{(1)} & c_3^{(1)} & & \\ & \ddots & a_2^{(1)} & b_3^{(1)} & c_4^{(1)} & \\ 0 & & \ddots & & \ddots & \end{bmatrix}\begin{bmatrix}L_0^{(1)}\\L_1^{(1)}\\L_2^{(1)}\\L_3^{(1)}\\\vdots\end{bmatrix}=\begin{bmatrix}d_0^{(1)}\\d_1^{(1)}\\d_2^{(1)}\\d_3^{(1)}\\\vdots\end{bmatrix}, \tag{25}$$

with

$$a_n^{(1)}\equiv-(n+1)\mathrm{e}^{-\left(n+\frac{1}{2}\right)\beta_1},\ n\geq0, \tag{26}$$

$$b_n^{(1)}\equiv\left[\sinh\beta_1+(2n+1)\cosh\beta_1\right]\mathrm{e}^{-\left(n+\frac{1}{2}\right)\beta_1},\ n\geq0, \tag{27}$$

$$c_n^{(1)}\equiv-n\mathrm{e}^{-\left(n+\frac{1}{2}\right)\beta_1},\ n\geq1, \tag{28}$$

$$d_n^{(1)}\equiv2\left[-n\mathrm{e}^{\left(n-\frac{1}{2}\right)\beta_1}C_{n-1}^{(1)}+(2n+1)\cosh\beta_1\mathrm{e}^{\left(n+\frac{1}{2}\right)\beta_1}C_n^{(1)}-(n+1)\mathrm{e}^{\left(n+\frac{1}{2}\right)\beta_1}C_{n+1}^{(1)}\right],\ n\geq0. \tag{29}$$

$C_n^{(1)}$ are known from the solution of the outside field problem (see (7) and (15), (16)) and $L_n^{(1)}$ are calculated by truncating appropriately (25) in terms of the accuracy required.

*Inside the sphere* $\beta=\beta_2$ the scalar potential has the form

$$\Psi_i^{(2)}(\alpha,\beta)=(\cosh\beta-\cos\alpha)^{1/2}\times\sum_{n=0}^{\infty}L_n^{(2)}\mathrm{e}^{-\left(n+\frac{1}{2}\right)\beta}P_n(\cos\alpha). \tag{30}$$

$L_n^{(2)}$ are derived from the continuity condition

$$\left(\frac{\partial\Psi_i^{(2)}}{\partial n}\right)_{\beta=\beta_2}=\left(\frac{\partial\Psi}{\partial n}\right)_{\beta=\beta_2}. \tag{31}$$

The same procedure is employed to determine the coefficients $L_n^{(2)}$ which are obtained from (25) with the superscripts (1) changed in (2) and with

$$a_n^{(2)} \equiv (n+1)\mathrm{e}^{-\left(n+\frac{1}{2}\right)\beta_2}, \; n \geq 0, \tag{32}$$

$$b_n^{(2)} \equiv \left[\sinh\beta_2 - (2n+1)\cosh\beta_2\right]\mathrm{e}^{-\left(n+\frac{1}{2}\right)\beta_2}, \; n \geq 0, \tag{33}$$

$$c_n^{(2)} \equiv n\mathrm{e}^{-\left(n+\frac{1}{2}\right)\beta_2}, \; n \geq 1, \tag{34}$$

$$\begin{aligned} d_n^{(2)} \equiv 2\Bigg[ & -n\mathrm{e}^{\left(n-\frac{1}{2}\right)\beta_2} C_{n-1}^{(2)} \\ & + (2n+1)\cosh\beta_2 \mathrm{e}^{\left(n+\frac{1}{2}\right)\beta_2} C_n^{(2)} \\ & - (n+1)\mathrm{e}^{\left(n+\frac{1}{2}\right)\beta_2} C_{n+1}^{(2)}\Bigg], \; n \geq 0, \end{aligned} \tag{35}$$

It should be noted that, while the potential $\Psi$ defined outside the spheres is constant over the two sphere surfaces ($\Psi_1$ and $\Psi_2$, respectively), the potentials $\Psi_i^{(1)}$ and $\Psi_i^{(2)}$ inside the two spheres are not constant over their respective surfaces.

Numerical results for the magnetic flux density at selected points inside the two spheres are in good agreement with those in [3].

## 3. CONCLUSIONS

For an arbitrary direction of the uniform magnetic field with respect to the common axis of the spheres, the solution is derived by taking separately the field components along the *z*–axis (as shown above) and along the *x*–axis. The potentials for the latter component are expressed in terms of associated Legendre functions $P_n^1(\cos\alpha)$ and have an azimuthal $\cos\varphi$ dependence.

As in the case of uncharged conducting spheres in an axial electric field, there is a substantial intensification of the magnetic field in the gap between the two ferromagnetic spheres in an uniform $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$, the flux density at the points A and B in Fig. 1 becoming hundreds of times greater then $B_0$ when the separation between spheres becomes hundreds of times smaller than their radii. Correspondingly, the magnetic flux density increases substantially at points close to A and B inside the spheres, its value decreasing to the level of $B_0$ for points far from A and B. For small values of $B_0$ the actual material inside the spheres remains linear, but for large external field intensities the material close to the points A and B may become magnetically saturated.

Since only tridiagonal matrix equations are involved, the method presented is substantially more efficient than analytic methods previously published which involve dense matrix equations.

Using the procedure in this paper, solutions of a controllable accuracy can also be developed for other Laplacian fields in the presence of two spheres, for instance for dielectric spheres in electric fields and for linear magnetic spheres of finite permeability in magnetic fields.

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