THE ENERGETIC PROPERTIES OF THE ELECTROMAGNETIC FIELD

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The aim of the paper is the analysis of relations that express the energy of the electromagnetic field of a domain bounded by a closed surface placed inside a body. There is considered that the surface which bounds the domain has on its two faces, outwards and inwards, surface densities of the electric polarization charge and of the magnetization current sheet, equal in absolute value but of opposite sign. If the terms of the balance equations are verified, the sources of the internal surface have a significant importance in the computing relationships.

1. INTRODUCTION

A microscopic electrically charged particle, alone in the whole space and the electromagnetic field associated to it, and produced by it, form an inseparable whole, a single physical system. The energy of the system particle-field and the momentum (quantity of motion or momentum extended to the whole infinite space summed up with the mechanical momentum of the particle) of the same physical system are conserved. A domain bounded by a closed surface which contains the particle is a domain of the physical system substance-field that include the two strictly connected two components, the substance (the particle) and the electromagnetic field. A space domain that does not contain the particle at the considered moment is a domain that contains a single component of the physical system, namely electromagnetic field.

Two electrically charged particles, single in space interact each other, at a distance (also called actio in distance), by the propagation of the field associated to them, in the contiguous medium parts. For instance, the electromagnetic field of the first particle acts on the substance of the second particle. However, in the treatment of the present analysis, the field of each of the two particles is not

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considered as being two *distinct* physical *interacting* systems, but the components of a single physical system, namely the substance (of both particles)-field.

Similarly, one will assume that several particles, and the field associated to them, form a single physical system extended to the whole space. The interaction between the parts of the system substance-field, where by interaction it is meant the exchanges of energy and momentum, will be examined.

Generally, in this analysis, we shall consider that a body and the electromagnetic field in that it is situated form a single physical system, in which the two components of the system, the substance of the body and the electromagnetic field are connected and are also interacting. The temporary electromagnetic states of the substance (electrokinetic, electrification and magnetization) are due to the presence of the field within the substance, and the field state quantities are depending on these states. A domain of the body, bounded by a closed surface contained by the body, is a part of the physical system. We shall express the electromagnetic energy of the domain, and also the energy associated with the electromagnetic field and the electromagnetic energy associated with the substance. Similarly, we shall express the variation with time of the electromagnetic energy and the flux (flow) of energy of the global system and of the electromagnetic field, which is a component of the physical system of the considered domain.

No condition concerns the material electromagnetic properties of the body substances, the bodies may be at rest or in motion, the general case of non-stationary condition will be considered (i.e., generally variable with time). No material law will be considered, in order to keep the general character of the corresponding Maxwell equations. In proofs, only the theory of the electromagnetic quantities as it was systematically formulated by Remus Răduleț [5-7] and the local forms of the general laws of the electromagnetic phenomena (the Maxwell equations in the case of domains fulfilling the conditions of continuity and smoothness, in their interpretation given by Minkowski [3], hence in the form corresponding also to the case of moving media) will be used.

The energetic properties of the system substance-field, may be deduced by means of Maxwell equations, in which occur the state field quantities, $E, D, B, H$, and the electromagnetic state quantities of the body, $\rho, J$, [8–14, 17–18]. The energetic properties of the system components can be deduced if in the Maxwell equations the substance is characterized by the state quantities $\rho, P, J$ and $M$ and the electromagnetic field by the state quantities $E$ and $B$, [1–4, 16–20].

As known from Thermodynamics, the work acting on a system of bodies in any electromagnetic state, results in an increase of the electromagnetic energy of the system and in the producing of a quantity of heat. By *free energy* is meant the amount of energy that can be converted into work. In the case of reversible processes, we shall assume that all transformations take place in the *condition of*
constant temperature. According to the previous condition, the free energy will be given by the internal energy of the system minus the quantity of heat delivered by the system to surrounding bodies at constant temperature [8, p. 182]. Consequently, the volume electromagnetic field density will contain only electromagnetic field state quantities. Therefore, the electromagnetic energy (also called internal electromagnetic energy) will coincide with the free energy of the system [8, p. 183].

2. GENERAL RELATIONS

The primitive electromagnetic state quantities of bodies: $q, p, i, m$, or their volume densities, $\rho_v, P, J$ and $M$ have been introduced by measuring the ponderomotive actions exerted, in the reference frame of the laboratory, in vacuo, by the electric and magnetic stationary fields, on small proof bodies (test bodies, check bodies), at limit material points, i.e., point-like bodies, being in various stationary electromagnetic states (electrification, electric polarization, electrokinetic, magnetization) [4–8].

The forces exerted on certain small test bodies (immobile with respect to the system of reference relatively to which the measurement are carried out) are expressed by certain formulae considered as experimentally established, for certain conditions. These formulae may be considered as corresponding to some thought experiments. In fact, these formulae were established as a generalization of several particular experiments, their application being not contradicted by any experimental result. Their expressions, [1-7], are given below:

\[
F = qE, \quad f \, dV = \rho_v, \, dV \, E, \tag{1 \ a, b}
\]

\[
F = \text{grad} \left( p \cdot \mathbf{E} \right), \quad f \, dV = \text{grad} \left( P \, dV \cdot \mathbf{E} \right), \tag{2 \ a, b}
\]

\[
dF = i \, dl \times B, \quad f \, dV = J \, dV \times B, \tag{3 \ a, b}
\]

\[
F = \text{grad} \left( m \cdot \mathbf{B} \right), \quad f \, dV = \text{grad} \left( M \, dV \cdot \mathbf{B} \right), \tag{4 \ a, b}
\]

where the arrow indicates the quantity that is differentiated.

One considers that the electromagnetic states of the test bodies are electromagnetic states of the substances of their volume. A small electrically polarized body, for instance, has superficially not anything distinctive, its
The electromagnetic state is a volume state. The volume densities of the forces will be calculated by the following relations:

\[ f = \rho_s E, \quad f = \nabla \cdot \left( \mathbf{P} \cdot \mathbf{E} \right), \quad f = J \times \mathbf{B}, \quad f = \nabla \times \left( \mathbf{M} \cdot \mathbf{B} \right). \] (5 a, b, c)

In literature, also other expressions are given for the calculation of forces, which act on certain electrically and magnetically polarized small bodies, situated in vacuo and in non-homogeneous fields. For instance, in [5, 8, 10, 12], the following expressions of forces are used:

\[ F = (\mathbf{p} \cdot \nabla)\mathbf{E}_v, \] (6)

\[ F = (\mathbf{m} \cdot \nabla)\mathbf{B}_v. \] (7)

In a stationary regime, a small electrically polarized body is equivalent from the point of view of its electromagnetic behaviour, with one ideal electric dipole, situated in vacuo. A small-magnetized body is equivalent to one ideal current loop, situated in vacuo. These computing models are associated to the substance of the infinitesimal volume elements of polarized or magnetized bodies. The density of the electric polarization charges and the densities of the amperian electric currents are:

\[ \rho_{pv} = - \text{div}_v \mathbf{P}, \quad \rho_{ps} = - \text{div}_s \mathbf{P}, \]

\[ \mathbf{J}_m = \text{curl}_v \mathbf{M}, \quad \mathbf{J}_{ms} = \text{curl}_s \mathbf{M}. \] (8 a, …, d)

If within a macroscopic theory, the electric and magnetic polarization states are volume state of the substance, in the computing models, the electric charges and the fictitious electric currents have volume distributions, as well as surface ones, within a domain where the electric permittivity and the magnetic permeability of the body substance change into the electric permittivity of vacuum, \( \varepsilon_0 \), and the magnetic permeability, \( \mu_0 \), of vacuum, respectively. Although the polarization electric charges and the amperian currents may be considered as fictitious quantities, upon their interaction with the electromagnetic field, there result computing quantities, namely forces and powers, which are in good agreement with the quantities measured in the laboratory system of reference.

The formulae of densities keep their validity also in the regime variable with time of the electromagnetic field. This statement will be proved below by the fact that the densities of the body electromagnetic state quantities occur in the fundamental energetic identity of electromagnetism. This identity is a direct consequence of the Maxwell equations.
The surface $\Sigma$ bounds a domain with continuity of a body situated in an electromagnetic field. Its importance in the proofs, which follow, is significant. The surface has two faces, the internal face that bounds the substance of the inside of the domain, and the external surface that bounds the substance of the outside of the domain.

3. THE ENERGETIC IDENTITY IN ELECTROMAGNETISM

Starting from the Maxwell equations, two identities may be derived, the energetic and the dynamic one, respectively, which are very important in the study of electromagnetic phenomena. The identities, like the local forms of the general laws, are relations that are true relationships at any time moment $t$, and at any point definite by the position vector $r$ in space, if this point is situated within a domain with continuity and smoothness of the electromagnetic properties of the material.

Let be a body situated within an electromagnetic field variable with time. At point $r$, situated within an infinitesimal volume element $dV$ of a domain with continuity within the body, the electromagnetic state of the substance is characterized, generally, by the quantities: volume density of the electric charge, $p_e(r,t)$, density of the conduction electric current $J(r,t)$, electric polarization $P(r,t)$ and magnetization $M(r,t)$. The electromagnetic field at point $r$ has the following state quantities: electric field strength $E(r,t)$, magnetic induction $B(r,t)$, magnetic field strength $H(r,t)$ and electric induction $D(r,t)$. The following two relationships express the relation between the local state electromagnetic field state quantities and the local electromagnetic material state quantities. These relationships are the following:

$$D(r,t) = \varepsilon_0 E(r,t) + P(r,t), \quad (9 \ a)$$

and

$$H(r,t) = \frac{1}{\mu_0} B(r,t) - M(r,t). \quad (9 \ b)$$

In the study of the interaction between the substance of the infinitesimal volume $dV$, and the field in which it is placed, the quantities electric induction $D$ and magnetic field strength $H$ are field state quantities that depend on quantities electric polarization $P$, and magnetization $M$, of the substance of the infinitesimal volume element $dV$.

In order to characterize the electromagnetic state of the physical system body plus field, at a point of position vector $r$ and at the time moment $t$, six quantities are necessary and sufficient. In the macroscopic electrodynamics (Maxwell,
Minkowski), one chooses the electromagnetic field state quantities of the set \{E, D, B, H\} to which, two \{\rho_v, J\} substance state quantities have to be added. By this choice, one considers that the substance and the field in which the substance is placed form a single physical system, because the state quantities D and H, although attributed to the electromagnetic field, are state quantities of the ensemble substance-field. The corresponding relationships are:

\[ D = \varepsilon_0 E + P, \]  
(10 a)

\[ H = \frac{1}{\mu_0} B - M. \]  
(10 b)

The electromagnetic state quantities of the field and substance are measured in a system of reference fixed to the substance. The electromagnetic quantities in the Maxwell and Minkowski equations are quantities defined in the laboratory system of reference, in which the co-ordinates of the position vector \( r \), and the time \( t \), are measured.

The general laws of the electromagnetic field, corresponding to those of the work of Maxwell [1, Art. 619], can be written in the simplest form as follows:

\[ \text{curl} E = -\frac{\partial B}{\partial t}, \]  
(11)

\[ \text{curl} H = J + \frac{\partial D}{\partial t}, \]  
(12)

\[ \text{div} D = \rho_v, \]  
(13)

\[ \text{div} B = 0. \]  
(14)

Relation (14), i.e., the local form of the law of the magnetic flux, may be considered like a consequence of relation (11), the divergence of \( B \) resulting as constant, if one chooses conventionally this constant as zero.

The system of equations above keeps the same form for any inertial system of reference. The passage from one inertial system of reference to another inertial system of reference in motion relative to the former can be performed by the Lorentz transformation formulae [2, No. 169], [12, p. 88, 126], [17, p. 198]. Correspondingly, the state quantities change, yielding relations called by certain authors Minkowski relations [17, p.198].

If the physical considered system (substance-field) is characterized by four local electromagnetic state quantities of the body: \( \rho_v(r, t), J(r, t), P(r, t), \)
$\mathbf{M}(r,t)$, and two field state quantities $\mathbf{E}(r,t)$ and $\mathbf{B}(r,t)$, the interaction substance-field can be studied more suggestively. We shall use the Maxwell equations in which only these two quantities appear, hence the equations with $\{\mathbf{E}, \mathbf{B}\}$.

By using relations (11)–(14) and (10 a, b), one obtains:

\[
\text{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (15)
\]

\[
\text{curl} \frac{1}{\mu_0} \mathbf{B} = \left( \mathbf{J} + \text{curl} \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \right) + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (16)
\]

\[
\text{div}(\epsilon_0 \mathbf{E}) = \rho_v - \text{div} \mathbf{P}, \quad (17)
\]

\[
\text{div} \mathbf{B} = 0. \quad (18)
\]

In the right-hand side of equation (16), there appear all sources of the magnetic induction, which are located within the body substance, and in the right-hand side of equation (17), there appear all sources of the electric field, which are located within the substance. The following symbols, in the extended sense, will be used in the Maxwell equations:

\[
\rho_e = \rho_v + \rho_{pv} = \text{div} \mathbf{D} - \text{div} \mathbf{P} = \text{div}(\epsilon_0 \mathbf{E}), \quad (19)
\]

\[
\mathbf{J}_e = \mathbf{J} + \mathbf{J}_m + \mathbf{J}_p = \mathbf{J} + \text{curl} \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}. \quad (20)
\]

Therefore, the equations of the macroscopic Electrodynamics will have the same form as the equations of the microscopic Electrodynamics:

\[
\text{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (21)
\]

\[
\text{curl} \frac{1}{\mu_0} \mathbf{B} = \mathbf{J}_e + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (22)
\]

\[
\text{div} \mathbf{E} = \frac{\rho_e}{\epsilon_0}, \quad (23)
\]

\[
\text{div} \mathbf{B} = 0. \quad (24)
\]
In the classical microscopic Electrodynamics (the theory of electrons), in which the microscopic quantities are used, the Lorentz equations [2, Ch. 1, No. 7, relations (17) – (20)], with certain slight modifications of symbols, are:

\[ \text{curl } e = \frac{\partial b}{\partial t}, \]  \hspace{1cm} (25)

\[ \text{curl} \frac{b}{\mu_0} = j + e_0 \frac{\partial e}{\partial t}, \]  \hspace{1cm} (26)

\[ \text{div } e = \frac{\rho_e}{\epsilon_0}, \]  \hspace{1cm} (27)

\[ \text{div } b = 0, \]  \hspace{1cm} (28)

to which the Lorentz law of ponderomotive action has to be added [2, Ch. 1, No. 7, relation (23)]:

\[ f = \rho_e e + j \times b. \]  \hspace{1cm} (29)

If beside the Maxwell equations (21) – (24), two force densities will be used, namely a volume density and a surface density, then the theorems of energy conservation, and of the momentum conservation, of the macroscopic Electrodynamics will have the same forms like the corresponding theorems of microscopic electrodynamics.

If one considers a closed surface \( \Sigma \) within a domain with continuity of the physical system substance-field, then on this surface with two faces \( \Sigma^- \) and \( \Sigma^+ \), the single density that could exist are the surface density of fictitious electric polarization charges and the densities of the amperian electric currents, because on the surface \( \Sigma \) (or on \( \Sigma^- \) and on \( \Sigma^+ \)), situated within a domain with continuity, the following relations are fulfilled:

\[ \text{div } D = 0, \text{ curl } H = 0. \]  \hspace{1cm} (30 a, b)

The mentioned densities are equal in absolute value and have opposite sign, and are given by relations:

\[ \rho_{\Sigma^-} = \rho_{\Sigma^+} = \mathbf{P} \cdot \mathbf{n}, \hspace{0.5cm} J_{\Sigma^-} = J_{\Sigma^+} = \mathbf{M} \times \mathbf{n}. \]  \hspace{1cm} (31 a, b)

These are the densities of certain sources, which belong to the substance of the physical considered system, bounded by the surface \( \Sigma \). Taking into account relations of the formula (29), but for the macroscopic case, and relations (19), (20), (31 a, b), the volume and surface densities of the force will be:
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\[ f_v = \rho_v E + J_v \times B = \left[ \text{div}(\varepsilon_0 E) \right] E + \left( J + \frac{\partial P}{\partial t} + \text{curl} M \right) \times B, \]  

\[ f_s = \rho_s \varepsilon_0 E + J_s \times B = (P \cdot n) E + (M \times n) \times B. \]  

(32 a, b)

Let equation (11) be multiplied as a scalar product by \( H \) and equation (12) by \( E \), and then subtracted. We shall obtain:

\[ E \cdot \left( J + \frac{\partial D}{\partial t} \right) + H \cdot \frac{\partial B}{\partial t} + \text{div}(E \times H) = 0. \]  

(33)

This relation valid at any point defined by position vector \( r \), of a domain with continuity, at any time moment \( t \), will be called the fundamental energetic identity in Electromagnetism. The terms of equation (33) will be multiplied by the infinitesimal volume \( dV \) about the point defined by \( r \). Similar calculation will be carried out for all volume elements in which the domain with continuity has been divided, and then the integration will be extended over the volume \( V \) of the domain. We shall obtain:

\[ \int_{V} \left[ E \cdot \left( J + \frac{\partial D}{\partial t} \right) + H \cdot \frac{\partial B}{\partial t} \right] dV + \oint_{\Sigma} (E \times H) \cdot n dS = 0. \]  

(34)

It is interesting to look for a physical interpretation of this mathematical identity. From the beginning, there is no reason to consider the relation like a balance of power and to call it energetic identity. If this property was justified, the relation would be very important because being obtained from general laws, it will keep the independence of the nature of material properties, the regime of the electromagnetic field (i.e., variation with time) and the state of motion of bodies. For linear, homogeneous, isotropic and immobile media, the volume integral of relation (34) is a known value. Indeed, from the study of particular regimes of the electromagnetic field, it is known that the mentioned integral represents the derivative with respect to time of the electromagnetic energy of the system substance-field of the domain bounded by surface \( \Sigma \) summed up with the electromagnetic power necessary for the electric conduction process. We have:

\[ \int_{V} \left[ E \cdot \left( J + \frac{\partial D}{\partial t} \right) + H \cdot \frac{\partial B}{\partial t} \right] dV = \int_{V} \frac{\partial}{\partial t} \left( \frac{1}{2} \rho E^2 + \frac{1}{2\mu} B^2 \right) dV + \oint_{\Sigma} E \cdot J dS. \]  

(35)

We shall assume that, generally, whatever be the material properties of the substance of the domain, and whatever be the motion state of the bodies, the time...
integral, from an initial state \( S_0 \{ E_0, D_0, B_0, H_0, J_0 \} \) of the physical system substance-field, to a final state \( S \{ E, D, B, H, J \} \), represents the electromagnetic energy of this physical system, accumulated from the initial moment, to moment \( t \). We shall obtain:

\[
W_{V_{\Sigma}} = \int_0^t \int_{V_{\Sigma}} \left[ E \cdot \left( J + \frac{\partial D}{\partial t} \right) + H \cdot \frac{\partial B}{\partial t} \right] \, dV.
\] (36)

The expression (36) is called the *energy integral of the electromagnetic field* [11, p. 274], and the expression without the term containing \( J \), is called the *internal energy of the electromagnetic field* [11, p. 275], simpler called *energy of the electromagnetic field* [9, 12, 17]. It is worth noting that whatever be the mentioned type of mentioned energy, it represents an energy of the system substance-field, therefore of the whole considered as a physical system. A part of the energy integral of the electromagnetic field is necessary for the electric conduction in substance, and to the magnetization of the substance of the domain of volume \( V_{\Sigma} \).

In accordance with the equation of balance of power, which follows from the principle of conservation of energy, the decrease with time of the electromagnetic energy of volume \( V_{\Sigma} \) is due to the power changed through the surface \( \Sigma \):

\[
- \frac{dW_{V_{\Sigma}}}{dt} = P_{\Sigma}, \quad \text{(37 a)}
\]

or:

\[
\frac{dW_{V_{\Sigma}}}{dt} + P_{\Sigma} = 0. \quad \text{(37 b)}
\]

The last term of the energetic identity in the integral form (34) should represent the electromagnetic energy changed by the substance-field system of volume \( V_{\Sigma} \), through surface \( \Sigma \), with the external system, but also transformed into other forms of energy in the unit of time, even in the substance located on surface \( \Sigma \). The power transmitted outwards the bound surface will be:

\[
P_{\Sigma} = \oint_{\Sigma} (E \times H) \cdot n \, dS.
\] (38)

If the relationships (10 a, b) are used, the variation of the electromagnetic energy of the substance-field system, and the power transmitted outwards the surface will be:
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\[
\frac{dW_{\Sigma}}{dt} = \int_{\Sigma} \left[ E \cdot \left( J + \frac{\partial D}{\partial t} \right) + H \cdot \frac{\partial B}{\partial t} \right] \, dV =
\]

\[
= \int_{\Sigma} \left( \frac{\partial}{\partial t} \left( \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right) \right) \, dV + \int_{\Sigma} E \cdot J \, dV + \int_{\Sigma} \left( E \cdot \frac{\partial P}{\partial t} - M \cdot \frac{\partial B}{\partial t} \right) \, dV.
\]

But, expanding the expression \( \text{div} (E \times M) \), there follows:

\[
- \int_{\Sigma} \frac{\partial B}{\partial t} \, dV = \int_{\Sigma} E \cdot \text{curl} M \, dV + \int_{\Sigma} E \cdot (M \times n) \, dS,
\] (39 a)

and therefore:

\[
\int_{\Sigma} \left[ E \cdot \left( J + \frac{\partial D}{\partial t} \right) + H \cdot \frac{\partial B}{\partial t} \right] \, dV = \int_{\Sigma} \frac{\partial}{\partial t} \left( \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right) \, dV +
\]

\[
+ \int_{\Sigma} E \cdot \left( J + \frac{\partial P}{\partial t} + \text{curl} M \right) \, dV + \int_{\Sigma} E \cdot (M \times n) \, dS.
\] (40)

If the surface \( \Sigma \) is at rest in the laboratory system of reference the variation with time of the electromagnetic energy \( W_{\Sigma} \) of the substance-field system is equal to the variation with time of the energy of the electromagnetic field in the proper sense, given by the following relation:

\[
W_{\text{em}} = \int_{\Sigma} \frac{\partial}{\partial t} \left( \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right) \, dV,
\] (41)

to which it has to be added the electromagnetic power transformed into other forms due to the conduction, polarization and magnetization electric currents, distributed in volume and on surfaces, according to relations:

\[
P_{Jf} = \int_{\Sigma} E \cdot J \, dV, \quad P_{Jf} = \int_{\Sigma} E \cdot J \, dS.
\] (42 a, b)

It is to be noted that expression (41) represents only a part of the usual expression and namely the expression of the energy of the electromagnetic field in
the proper sense. The difference between the two expressions (of the internal energy and that in the proper sense) represents the energy of the electromagnetic state of the substance and depends explicitly on its material quantities. In literature, the particular case of certain dielectric bodies has been analysed, and the mentioned difference represents an energy of elastic nature [8, p. 178], [12, p. 237], although it results from a modification of the electromagnetic field quantities.

The variation of the energy integral of the electromagnetic field may be written in the form:

$$\frac{dW_{\text{em}}}{dt} = \frac{dW_{\text{em}}}{dt} + P_J + P_{\text{em}}. \quad (43)$$

The interior electromagnetic power changed through the surface $\Sigma$ of the domain with continuity is equal to the flux of energy in the proper sense (energy flow-rate) of the domain (with continuity), minus the energy delivered even in the surface $\Sigma$ corresponding to the magnetization electric current:

$$P_\Sigma = \oint (E \times H) \cdot n dS = \oint \left( E \times \frac{1}{\mu_0} B \right) \cdot n dS - \oint E \cdot (M \times n) dS =$$

$$= P_{\text{em}} - P_J. \quad (44)$$

With these decompositions, the fundamental energetic identity (33) or (34) will be transformed into the identity:

$$- \int \frac{\partial}{\partial t} \left( \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \frac{1}{\mu_0} B^2 \right) dV = \int E \cdot \left( J + \frac{\partial P}{\partial t} + \text{curl} M \right) dV +$$

$$\oint \left( E \times \frac{B}{\mu_0} \right) n dS, \quad (45)$$

or, for a surface $\Sigma$ fixed with respect the laboratory system of reference:

$$- \frac{dW_{\text{em}}}{dt} = P_J + P_{\text{em}}. \quad (46)$$

Relation (45) is just the energetic identity deduced from the Maxwell equations in terms of vectors $\{E, B\}$. Indeed, if we perform the scalar multiplication of equation (15) by $B/\mu_0$, equation (16) by $E$ and we deduct the obtained equations, we get the identity:
\[-\frac{\partial}{\partial t} \left( \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2 \mu_0} B^2 \right) = E \cdot \left( \mathbf{J} + \text{curl} \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \right) + \text{div} \left( \mathbf{E} \times \frac{1}{\mu_0} \mathbf{B} \right). \quad (47)\]

In integral form, we get anew the identity (46):

\[-\frac{d}{dt} \int_{V_x} w_{em} \, dV = \int_{V_x} E \cdot J \, dV + \oint_{\Sigma} \Pi_{em} \cdot n \, dS. \quad (48)\]

The decrease velocity of the energy of the electromagnetic field in the proper sense, of the considered domain with continuity, \(V_x\), is equal to the power delivered in the substance of \(V_x\) by the electric currents produced by the charge carriers in the substance, to which it has to be added the flux of power through the surface \(\Sigma\). The volume density of the energy of the electromagnetic field in the proper sense, within a domain with continuity, is:

\[w_{em} = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2 \mu_0} B^2, \quad (49)\]

and the vector of the flux of energy in the proper sense, through the bound surface, in the unit of time, is:

\[\Pi_{em} = E \times \frac{1}{\mu_0} B. \quad (50)\]

### 4. CONCLUSIONS

From the fundamental energetic identities (34) and (45), written in the integral form, for domains with continuity and smoothness, there follow the conclusions below:

1. The electromagnetic field in the proper sense cannot be separated from the substance if the state quantities \(D\) and \(H\) are used. The electromagnetic energy (also called electromagnetic internal energy of a volume \(V_x\) is formed of the energy of the electromagnetic field in the proper sense and of the electromagnetic energy of the substance. The volume density of the electromagnetic energy, like the flux of electromagnetic energy, in the case of a body situated in electromagnetic field, is depending on the electromagnetic material properties of the body substance. For instance, for a linear medium, the volume density of the electromagnetic energy (also called internal electromagnetic energy) is:

\[w = \frac{1}{2} \mathbf{E} \cdot \mathbf{D} + \frac{1}{2} \mathbf{H} \cdot \mathbf{B} = \left( \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2 \mu_0} B^2 \right) + \left( \frac{1}{2} \mathbf{E} \cdot \mathbf{P} - \frac{1}{2} \mathbf{B} \cdot \mathbf{M} \right), \quad (51)\]
and the density of the flux of electromagnetic energy is:

$$\Pi = E \times H = E \times \frac{1}{\mu_0} B - E \times M = \Pi_{em} - E \times M.$$  \hspace{1cm} (52)

2. The density of the electric current in the extended sense, within the substance of a body, in a domain with continuity, is the sum of the densities of the following components: conduction electric currents, amperian (magnetization) electric current, polarization electric current:

$$J_e = J + \text{curl } M + \frac{\partial P}{\partial t}.$$  \hspace{1cm} (53)

The density of the electric current sheet, located on the face $\Sigma$ of the surface $\Sigma$, situated in the continuity domain, is:

$$J_{\Sigma} = M \times n.$$  \hspace{1cm} (54)

The volume density of the energy of the electromagnetic field transferred to the substance in the unit of time is:

$$p_{J_e} = J_e \cdot E = \left( J + \text{curl } M + \frac{\partial P}{\partial t} \right) \cdot E,$$  \hspace{1cm} (55)

relation that is analogous to that of the electro-calorific effect for conductors, valid regardless of: the variation regime of the electromagnetic field, the electromagnetic properties of the substance, and the motion state of the substance. The volume density of the power irreversibly transformed into heat by electric conduction, and the terms having the physical dimensions of power density:

$$E \cdot \text{curl } M, \quad E \cdot \frac{\partial P}{\partial t},$$  \hspace{1cm} (56)

are related to the polarization and magnetization phenomena, which correspond to densities of power reversible in the case of linear media. The surface density of the term related to the electromagnetic energy is:

$$p_{J_e} = J_{\Sigma} \cdot E = (M \times n) \cdot E.$$  \hspace{1cm} (57)

3. The identity (34) derived from the Maxwell equations expressed in terms of the vector set $\{E, B, D, H\}$ represents the balance of electromagnetic power of the physical system substance-field, and the identity (45), which refers to the electromagnetic power in the proper sense, derived from the Maxwell equations, expressed in terms of the vector set $\{E, B\}$, represents the balance of the
The energetic properties of the electromagnetic field

Electromagnetic power of the field in the proper sense, a component of the physical substance-field system:

\[ \int \left[ \frac{\partial}{\partial t} \left( \frac{1}{2} \frac{\varepsilon_0 E^2}{\mu_0} + \frac{1}{2} B^2 \right) + E \cdot J_e \right] dV = \oint \left( E \times \frac{1}{\mu_0} B \right) \cdot n dS. \] (58)

4. The theorem of the electromagnetic energy in the theory of electrons has the integral form:

\[ \int \left[ \frac{\partial}{\partial t} \left( \frac{1}{2} \varepsilon_0 e^2 + \frac{1}{2} \frac{\mu_0 b^2}{\mu_0} \right) + e \cdot j \right] dV = \oint \left( e \times \frac{1}{\mu_0} b \right) \cdot n dS, \] (59)

of identical form to the theorem of the energy of the macroscopic electromagnetic field (58).

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3. H. Minkowski, *Space and Time*, an address delivered at the 80th Assembly of German Natural Scientists and Physicians (Gesellschaft Deutscher Naturforscher und Ärzte) at Cologne, September 21, 1908 (in German), referred to in [4].


