THE IMAGE CHARGES AND CAPACITANCE FOR TWO CONDUCTIVE SPHERES

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In the paper the method of Kelvin transform is used in an original way for determining the image charges and the capacitance for a device made of two conductive spheres situated at the same potential. The capacitance is calculated with respect to infinity. The situations, when the two spheres are exterior, are analyzed.

1. **INTRODUCTION**

In recent years, the method of the Kelvin transform has come back into actuality [1]. A great part of the applications involves field computation in dielectrics with rare metallic inclusions [2, 3]. Generally the field of two conducting spheres is studied [4, 5], in the presence of exterior field sources and often the case of orthogonal spheres is taken into consideration. In the present paper the authors apply the method of the Kelvin transform in conjunction with the method of the supplementary charge [6]. We have not found a similar approach in the consulted literature.

The principle of the method applied in the paper is presented.

We consider the surfaces S_k (k = 1, 2 ... n) which have the same potential V. P is picked as a inversion pole and p^2 as the power of inversion [7]. If the point $\mathrm{A} \in S_k$ and the potential of this point is $V_A = V$ then the inverse of the point A is $\mathrm{A}' \in S_k'$ and

$$\mathrm{PA}' = \frac{p^2}{\mathrm{PA}}, \tag{1}$$

with $\mathrm{A}' \in \mathrm{PA}$, and the value of the potential is (Kelvin transform):

$$V_{A'} = \frac{\mathrm{PA}}{p} V_A = \frac{\mathrm{PA}}{p} V \,. \tag{2}$$

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So the inverse surfaces $S_k^{/}(k = 1, 2, ... n)$ are not equipotential.

If in the P pole is placed a supplementary charge q_0 [6] where:

$$q_0 = -\ 4\pi\varepsilon_0 pV\ , \qquad (3)$$

then the potential of $\mathrm{A}^{/}$ point becomes $V_{A^{/}}^{/}$:

$$V_{A^{/}}^{/} = \frac{\mathrm{PA}}{p}V + \frac{q_0}{4\pi\varepsilon_0 \mathrm{PA}^{/}} = \frac{\mathrm{PA}}{p}V\ -\ \frac{p}{\mathrm{PA}^{/}}V = 0\ . \qquad (4)$$

So the surfaces $S_k^{/}(k\ =\ 1, 2, ... n)$ become equipotential, having the potential zero.

We now determine the image charges $q_i(i\ =\ 1, 2, ... m)$ of the supplementary charge q_0 related to the $S_k^{/}(k = 1,2... n)$ surfaces of zero potential. These charges will be called primary image charges.

A new inversion with the same pole P and the same power p^2 is applied.

The surfaces $S_k^{/}(k = 1, 2, ... n)$ will be transformed back into the initial surfaces S_k (k = 1, 2, ... n). The primary image charges $q_i\,(i = 1, 2, ... m)$, situated in the points $\mathrm{M_i}$ are being transformed into the final image charges $q_i^{/}(i = 1, 2, ... m)$ situated in the $\mathrm{M}_i^{/}$ points where $\mathrm{PM}_i^{/} = \dfrac{p^2}{\mathrm{PM}_i}$ with $\mathrm{M}_i^{/} \in \mathrm{PM}_i$ and [8]:

$$q_i^{/} = \frac{p}{\mathrm{PM}_i}q_i\,. \qquad (5)$$

The final image charges $q_i^{/}$ without the charge q_0 from P, which is eliminated, will establish the V potential in the points of the $S_k(k\ =\ 1,\ 2\ ...\ n)$ surfaces.

We shall present an exemplification for the method for the particular case $n = 1$, the considered surface S_1 being a sphere at the given potential V, with the radius a (Fig. 1). We wish to find the position and value of the image charge. We mention that in the explanatory Fig. 1–Fig. 6, the real surfaces are drawn with a full line and the fictional ones are drawn with an interrupted one.

The previously presented stages are followed. The pole P is taken on the sphere and the power of the inversion $p^2 = 4a^2$ so that the sphere will be transformed into a plane that is tangent to the given sphere, a plane whose potential varies according to the distance from the point on the plane to the pole (Fig. 2).

Through the introduction of the supplementary charge (3) in the pole

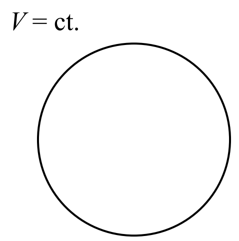

Fig. 1 – The sphere of a radius with constant potential whose electric image is searched for.

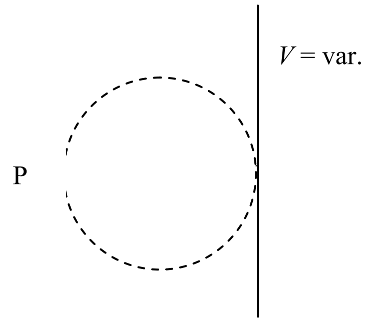

Fig. 2 – The transformation of the sphere in a plane with a potential that varies according to the distance to the pole P ($p^2=4a^2$).

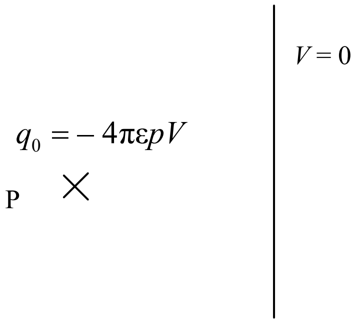

Fig. 3 – The introduction of the supplementary charge in the pole so that the plane potential is zero.

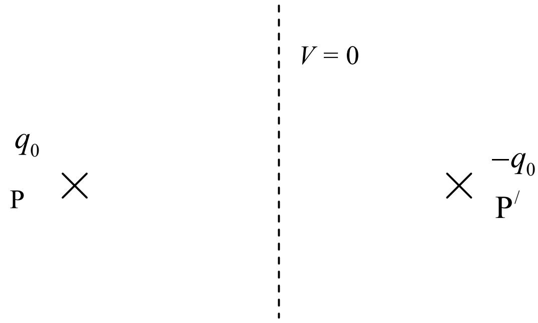

Fig. 4 – The introduction of the primary image charge – q_0 so that the plane with zero potential becomes fictional.

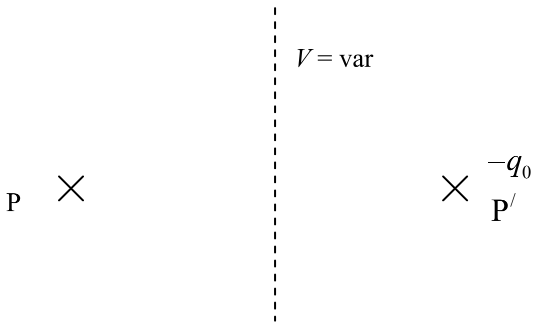

Fig. 5 – The elimination of the supplementary charge q_0 from the pole so that the potential on the fictional plane given by the charge from P comes back to the value on the real plane from Fig. 2.

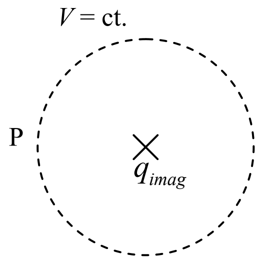

Fig. 6 – The transformation of the fictional plane from Fig. 5 into the fictional sphere and of the primary image charge $-q_0$ into the wanted image charge $q_{imag.} = 4\pi\varepsilon a V$.

Fig. 3), the plane becomes of potential zero (4). Now the primary image charge is introduced (Fig. 4) so that the plane with zero potential becomes fictional.

Through the elimination of the supplementary charge in the pole (Fig. 5), the potential created by the primary image charge in every point on the surface of the fictional plane is equal to the potential of the real plane from Fig. 2. A final inversion with the same pole and power (Fig. 6) transforms the fictional plane into the fictional sphere with the radius *a* and the primary image charge into the wanted image charge of the sphere. The value of this image charge is computed using formula (5) and is $q_{imag.} = -q_0 \dfrac{p}{4a} = 4\pi\varepsilon a V$, *i.e.* exactly the expected value.

2. **THE IMAGE CHARGES FOR A SYSTEM OF TWO EXTERIOR CONDUCTING SPHERES SITUATED AT THE SAME POTENTIAL**

We consider the conductor spheres of O_1 and O_2 centre and radius R_1 and R_2 respectively (Fig. 7) situated at a distance d where $d > R_1 + R_2$. We consider that R_1, R_2, d and the potential V of spheres are known. We propose to find the position and the value of the image charges in dependence of R_1, R_2, d and V and to calculate the two conductor spheres' capacitance with respect to infinity.

2.1. THE ESTABLISHMENT OF THE MODIFIED INVERSE CONFIGURATION

The O*x* axis is the line O_1O_2 and the O*y* axis passes through O_1. The equation of the radical plane of the spheres is:

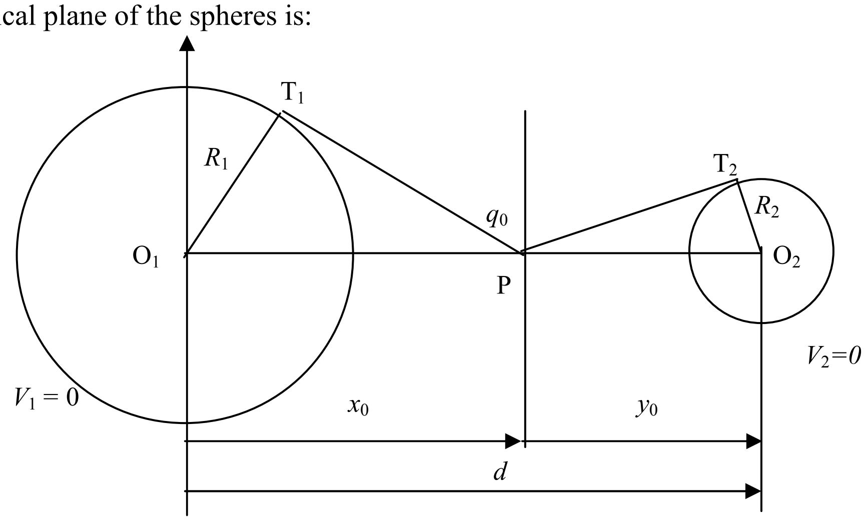

Fig. 7 – The "modified inverse configuration", with zero potential spheres and q_0.

$$x_0=[(R_1)^2-(R_2)^2+d^2](2d)^{-1}. \tag{6}$$

This plane intersects the drawing plane of Fig. 7 in P so $\mathrm{O_1P}=x_0$. Obviously $\mathrm{O_2P} = = y_0 = d-x_0$. The P point is situated on the radical axis so $\mathrm{PT_1}=\mathrm{PT_2}$.

This P point is picked as an inversion pole and $p^2=\mathrm{PT}_1^2$ as the power of inversion. This way the ($\mathrm{O_1}$, R_1) sphere transforms in itself [6], and likewise the ($\mathrm{O_2}$, R_2) sphere. After the inversion, the spheres' surfaces are not equipotential.

We now put the supplementary charge q_0 (3) in the P point of "inverse configuration". This new situation will be called "modified inverse configuration". In the "modified inverse configuration" the spheres become equipotential but with zero potential (Fig. 7). We now put the image charges related to the supplementary charge q_0. These charges will be called the "primary image charges".

2.2. THE PRIMARY IMAGE CHARGES IN MODIFIED INVERSE CONFIGURATION

The image charge of the q charge situated in the point M related to a zero potential sphere having O centre and R radius, M exterior of sphere, is [8]:

$$q^{/}=-\frac{R}{\mathrm{OM}}q\,, \tag{7}$$

situated in $\mathrm{M}^{/}$ point in OM line so that:

$$\mathrm{OM}^{/}=\frac{R^2}{\mathrm{OM}}\,. \tag{8}$$

The problem is complicated because we have (Fig. 7) two spheres with zero potential. Two groups of primary image charges exist.

a) The group, which will be called "first at left" and will be formed by:

q_{2k+1}, the image charge of the q_{2k} charge related to a ($\mathrm{O_1}$, R_1) sphere situated in the point A_{2k+1};

q_{2k+2}, the image charge of the q_{2k+1} charge related to a ($\mathrm{O_2}$, R_2) sphere situated in the point A_{2k+2};

b) The group which will be called "first at right" and will be formed by:

Q_{2k+1}, the image charge of the Q_{2k} charge related a ($\mathrm{O_2}$, R_2) sphere situated in the point B_{2k+1};

Q_{2k+2}, the image charge of the Q_{2k+1} charge related a ($\mathrm{O_1}$, R_1) sphere situated in the point B_{2k+2}, where k = 0, 1, 2....

a) For the position of "first at left" primary image charges the (8) formula is used.

The position related to the sphere (O_1, R_1) is with odd subscript and the one related to the sphere (O_2, R_2) is with even subscript. So $O_1A_{2k+1} = a_{2k+1}$ and $O_2A_{2k+2} = a_{2k+2}$.

With $a_0 = y_0$, and mathematical induction result for $k = 0, 1, 2, ...$:

$$a_{2k+1} = \frac{R_1^2}{d - a_{2k}} \qquad a_{2k+2} = \frac{R_2^2}{d - a_{2k+1}}. \tag{9}$$

It is shown that the arrays a_{2k+1} and a_{2k} are convergent and that $\lim_{k\to\infty} a_{2k+1} = a_{1\min} = x_0 - p$ and $\lim_{k\to\infty} a_{2k} = a_{2\min} = y_0 - p$.

The value of the primary image charges was obtained with (7) and mathematical induction:

$$q_{2k+1} = -\frac{a_1 a_2 ... a_{2k+1}}{R_1^{k+1} R_2^k} q_0 \qquad q_{2k+2} = \frac{a_1 a_2 ... a_{2k+2}}{R_1^{k+1} R_2^{k+1}} q_0 . \tag{10}$$

b) The position and the value of the primary image charges “first at right”was determined likewise.

2.3. THE RETURN TO THE „INITIAL CONFIGURATION”. THE FINAL IMAGE CHARGES

In the “modified inverse configuration” we’ll make a new inversion to the same pole P and the same power p^2. The supplementary charge will be eliminated. The spheres’ surface will be transformed back into the initial spheres’ surface. The primary image charges will be transformed into the final image charges. The inverse of the point H will be named H'. If for the points at the left of P we have $O_1H = h$ then $O_1 H'$ will be h'. If for the points at the right of P we have $O_2H = h$ then $O_2 H'$ will be h'. If the primary image charge of the point H is q then in the H' inverse point the final image charge will be q'.

The position and the value of the final image charges from the sphere (O_1, R_1).

From the “first at left” charges group we have in the (O_1, R_1) sphere:

$$PA'_{2k+1} = \frac{p^2}{PA_{2k+1}} = \frac{p^2}{x_0 - a_{2k+1}}; \qquad a'_{2k+1} = O_1A'_{2k+1} = x_0 - \frac{p^2}{x_0 - a_{2k+1}}. \tag{11}$$

With (5) we find the final image charges:

$$q'_{2k+1} = q_{2k+1}\frac{p}{PA_{2k+1}} = -\frac{a_1 a_2 ... a_{2k+1}}{R_1^{k+1} R_2^k} q_0 \frac{p}{x_0 - a_{2k+1}}. \tag{12}$$

From the “first at right” charges group we have in the (O_1, R_1) sphere:

$$\mathrm{PB}'_{2k+2}=\frac{p^2}{\mathrm{PB}_{2k+2}}=\frac{p^2}{x_0-b_{2k+2}}\,; \qquad b'_{2k+2}=\mathrm{O_1B}'_{2k+2}=x_0-\frac{p^2}{x_0-b_{2k+2}}\,. \tag{13}$$

The final image charges result:

$$Q'_{2k+2}=Q_{2k+2}\frac{p}{\mathrm{PB}_{2k+2}}=\frac{b_1b_2...b_{2k+2}}{R_1^{k+1}R_2^{k+1}}q_0\frac{p}{x_0-b_{2k+2}}\,. \tag{14}$$

The position and the value of the final images charges from the sphere (O_2, R_2) was determined like for the (O_1, R_1) sphere.

3. THE CAPACITY OF THE DEVICE MADE FOR THE TWO SPHERES

With the definition of the capacity formula:

$$\begin{aligned} C=4\pi\varepsilon_0 p^2\sum_{k=0}^{n}\Big(&\frac{a_1a_2...a_{2k+1}}{R_1^{k+1}R_2^{k}}\frac{1}{x_0-a_{2k+1}}-\frac{b_1b_2...b_{2k+2}}{R_1^{k+1}R_2^{k+1}}\frac{1}{x_0-b_{2k+2}}-\\ &-\frac{a_1a_2...a_{2k+2}}{R_1^{k+1}R_2^{k+1}}\frac{1}{y_0-a_{2k+2}}+\frac{b_1b_2...b_{2k+1}}{R_1^{k}R_2^{k+1}}\frac{1}{y_0-b_{2k+1}}\Big)\end{aligned}\,. \tag{15}$$

Theoretically n is infinite. Practically at every step k we verify the condition $\max\{a_{2k+1},b_{2k+2}\}-(x_0-p)<\mathrm{er}$, respectively $\max\{a_{2k+2},b_{2k+1}\}-(y_0-p)<\mathrm{er}$. The first k which verifies both conditions is the value of the final n.

It is easier to use the related capacity $C'=\frac{C}{C_{\max}}$, where $C_{max}=4\pi\varepsilon_0(R_1+R_2)$.

We shall demonstrate the convergence of formula (15). It will suffice to demonstrate the existence of a single infinite sum, out of the four sums involved. We will show that the series sum $\sum_{k=1}^{n}\left|q'_{2k+1}\right|$ exists and is finite when n tends to infinity. Using the fact that arrays a_{2k+1} and a_{2k} are decreasing we have that

$$\left|q'_{2k+1}\right|=\frac{a_1a_2...a_{2k+1}}{R_1^{k+1}R_2^{k}}\left|q_0\right|\frac{p}{x_0-a_1}<\left|q_0\right|\frac{pa_1}{R_1\left(x_0-a_1\right)}\left(\frac{a_1}{R_1}\right)^k\left(\frac{a_2}{R_2}\right)^k. \tag{16}$$

With (9) we have

$$\left|q'_{2k+1}\right| < \left|q_0\right| \frac{pa_1}{R_1\left(x_0 - a_1\right)} \left(\frac{R_1 R_2}{x_0 d - R_1^2}\right)^k . \tag{17}$$

It remains to demonstrate that $w = \dfrac{R_1 R_2}{x_0 d - R_1^2} < 1$.

For the demonstration we note $x_0 - R_1 = u$ and $d = R_1 + u + v + R_2$ from which we have $(R_1+u)(R_1+u+v+R_2) - R_1^2 = R_1R_2 + R_1(2u+v) > R_1R_2$ with equality for $u = v = 0$, a case in which the spheres are tangent. Then the necessary condition for convergence is accomplished because the terms of the serie tend to zero.

Then the bounding remains to be demonstrated:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left|q'_{2k+1}\right| < \frac{\left|q_0\right| pa_1}{R_1\left(x_0 - a_1\right)} \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{R_1 R_2}{x_0 d - R_1^2}\right)^k = \frac{\left|q_0\right| pa_1}{R_1\left(x_0 - a_1\right)} \frac{w}{1-w} . \tag{18}$$

which proves the convergence of the infinite sum.

The potential created by these charges in the exterior of the spheres is given by an infinite sum whose convergence results immediately from the convergence which was previously demonstrated.

The situation when the two spheres are tangent is the case limit for the previously considered situation, when the distance d between the centers of the spheres becomes very close to the sum of the radiuses R_1+R_2. On the other hand it can be considered a case limit for the problem from [9], where the case of the spheres which intersect under the angle $\varphi = \dfrac{\pi}{n}$ was considered. Through a computation similar to the one in the general case, we find:

$$C = 4\pi\varepsilon_0 \frac{R_1 R_2}{R_1 + R_2} \sum_{m=1}^{\infty} \left(\frac{1}{m - \dfrac{R_1}{R_1 + R_2}} + \frac{1}{m - \dfrac{R_2}{R_1 + R_2}} - \frac{2}{m} \right). \tag{19}$$

The convergence of the serie can be shown starting from Cauchy's formula $\dfrac{\Gamma'(1-a)}{\Gamma(1-a)} + \gamma = \sum_{m=1}^{\infty} \left(\dfrac{1}{m} - \dfrac{1}{m-a} \right)$, where Γ is Euler's Gamma function and γ is Euler's constant, $\gamma = 0.57721$. (We owe this observation to prof. math. dr. Mircea Ivan.)

4. **COMPUTATION EXAMPLE**

The values of the spheres' radius R_1 = 2 m, and R_2 = 1 m, and the distance between the centers of the spheres d = 6 m are given. The common potential of the two spheres is V = 1,000 V. We must find the position and the values of the final image charges and the capacitance of the system formed by the two spheres.

For the computation, MathCAD is used. The results of the distances are computed in meters and the results of the charges are computed in coulombs.

We obtain: $x_0 = 3.25$, $y_0 = 2.75$, $p = 2.562$

The final image charges in the (O_1, R_1) sphere are: $q_1^{/} = 2.222\ 10^{-7}$ situated in the centre O_1 of the sphere. $Q_2^{/} = -3.708\ 10^{-8}$ situated at a distance $b_2^{/} = 0.667$ from the centre O_1. $q_3^{/} = 1.271\ 10^{-8}$ situated at a distance $a_3^{/} = 0.686$ from the centre O_1. $q_4 = -1.68414\ 10^{-9}$ situated at a distance $a_4 = 0.688$ from the centre O_1. The q_4 charge contains five image charges, all situated at the distance $a_4 = x_0 - p = 3.25 - 2.562 = 0.688$ from the centre O_1.

The final image charges in the (O_2, R_2) sphere are $Q_1^{/} = 1.111\ 10^{-7}$ situated in the centre O_2 of the sphere. $q_2^{/} = -3.702\ 10^{-8}$ situated at a distance $a_2^{/} = 0.167$ from the centre O_2. $Q_3^{/} = 6.955\ 10^{-9}$ situated at a distance $b_3^{/} = 0.187$ from the centre O_2. $Q_4 = -2.074076\ 10^{-9}$ situated at a distance $b_4 = 0.188$ from the centre O_2. The Q_4 charge contains five image charges, all situated at the distance $b_4 = y_0 - p = 2.75 - 2.562 = 0.188$ from the centre O_2.

All the neglected image charges do not exceed 10^{-11}. The related capacity is $C^{/} = 0.825$ so the real capacity is $C = C^{/}\ 4\pi\varepsilon(R_1 + R_2) = 0.275\ 10^{-9}$ F.

The following results were obtained using a C++ program.

The value n was obtained with the presented conditions with er = 10^{-3}.

It is interesting to watch the variation of the related capacity when the distance d between the spheres takes different values. For R_1 = 2 m, R_2 = 1 m the pairs ($d \ldots C^{/}$) are:

(10000...0.9998), (1000...0.998), (100...0.986), (10...0.884), (8...0.861), (6...0.825), (4...0.771), (3.8...0.763), (3.6...0.756), (3.4...0.748), (3.2...0.742) (3.1...0.739).

For d between 3.020 and 3.013 we obtain the same value $C^{/} = 0.732$. This is obviously the value for the capacity for which the spheres can be considered tangent, and for $d \leq 3.012$ the program doesn't give any values.

In [9] the formula for the capacity of two conductor spheres which intersect under the angle $\frac{\pi}{n}$ was established. It is interesting to follow the variation of the related capacity of the device formed by the two spheres of radius R_1 = 2 m and R_2 = 1 m and depending on the n value, which was calculated like in the paper mentioned. With the notation $C^{/}(n)$ we have: $C^{/}(2)$ = 0.702, $C^{/}(3)$ = 0.718, $C^{/}(4)$ = 0.724, $C^{/}(5)$ = 0.727, $C^{/}(6)$ = 0.729, $C^{/}(7) = C^{/}(8)$ = 0.730, $C^{/}(9) = C^{/}(10) = C^{/}(11) = 0.731$, and $C^{/}(n \geq 12) = 0.732$.

The limit value 0.732 of the related capacity corresponds to the situation when the spheres can be considered tangent and is the same with the one found in our paper. The related capacity calculated with (19) is $C^{/}$ = 0.732 with three exact decimals, a result which was obtained for $m \geq 245$.

5. **CONCLUSIONS**

The formulas that were found are easy to implement using a mathematical utility program or a program in any programming language. The relations deducted for the image charges can be used for the field and potential computation in any point at the exterior of the spheres and for computing the charge density in any point of the spheres. The authors have found applications of the formulas established in the paper in the domains of earth electrodes and electrostatic separators. The results of this paper are a continuation of the results of paper [9].

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