A NEW PARTIAL DIFFERENTIAL EQUATION-BASED APPROACH FOR 3D DATA DENOISING AND EDGE PRESERVING

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This paper focuses on the denoising and enhancing of 3D data. Our aim is to denoise the 3D blocks without blurring relevant details such as edges. In order to achieve this task, we propose a new 3D tensor diffusion process based on the classical Coherence Enhancement Diffusion (CED) proposed by Weickert. The framework relies on the computation of a tensor providing information on the local orientation of structures. Then, the process is steered according to this directional information. In addition, the smoothing effect is adjusted according to a confidence measure that takes into account the regularity of the local structure. Through applications samples we will show the efficiency of our method on synthesized 3D data.

1. INTRODUCTION

Among the different methods to achieve the denoising of images, a large number of approaches using non-linear diffusion techniques have been proposed in the recent years. More often used in 2-dimensional cases, these methods are scalable for n-dimensional data. These techniques are based on the use of Partial Differential Equations (PDE).

The simplest diffusion process is the linear and isotropic diffusion:

\[
\frac{\partial U}{\partial t} = c \Delta U = \text{div}(c \nabla U).
\] (1)

When \( c \) is a constant, (1) is equivalent to a convolution with a Gaussian kernel. The similarity between such a convolution and the heat equation was proved by Koenderink [1].

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In this PDE, $U$ represents the intensity function of the data; $c$ is a constant which, together with the scale of observation $t$, governs the amount of isotropic smoothing. Setting $c = 1$, (1) is equivalent to convolving the image with a Gaussian kernel of width $\sqrt{2t}$. div indicates the divergence operator.

Nevertheless, the application of this linear filter over an image produces undesirable results, such as edge and relevant details blurring.

To overcome these drawbacks Perona and Malik [2] proposed the first non-linear filter by replacing the constant $c$ with a decreasing function of the gradient norms computed on the gray levels of the input image. This function plays the role of an edge detector that inhibits diffusion across edges and favors intra-region smoothing. Despite the quite convincing practical results, certain drawbacks remain unsolved: staircase effect [3] or pinhole effect [4] are often associated with the Perona Malik process. In addition, in the strongly noised regions, the model may enhance the noise. Since the introduction of this first non-linear filter, related works attempted to improve it [5].

Weickert [6, 7] proposed two original models with tensor based diffusion functions. The purpose of a tensor based approach is to steer the smoothing process according to the directional information contained in the image structure. The general model is written in PDE form, as:

$$\frac{\partial U}{\partial t} = \text{div}(D \nabla U),$$

with some initial and reflecting boundary conditions.

In the Edge Enhancing Diffusion (EED) model, the matrix $D$ depends continuously on the gradient of a Gaussian-smoothed version of the image ($\nabla U_\sigma$), where $\sigma$ determines the width of the convolution kernel. The aim of this Gaussian regularization is to reduce the noise influence, having as result a robust descriptor of the image structure. For 2D application, the diffusion tensor $D$ is constructed by defining the eigenvectors ($v_1$) and ($v_2$) according to $v_1 = \frac{\nabla U_\sigma}{||\nabla U_\sigma||}$ and $v_2 = \perp \nabla U_\sigma$ [6]. The corresponding eigenvalues $\lambda_1$, $\lambda_2$ were chosen in order to smooth only along edges and to inhibit diffusion across them.

Besides the EED model which enhances edges, Weickert proposed also a model for enhancing flow-like patterns: the Coherence Enhancing Diffusion – CED [8]. The structure tensor introduced in this model is a powerful tool for analyzing coherence structures. This tensor $J_\rho$ is able to measure the gradient changes within the neighbourhood of any investigated point:

$$J_\rho (\nabla U_\sigma) = K_\rho \ast (\nabla U_\sigma \otimes \nabla U_\sigma).$$
$U_\sigma$ is a smoothed version of the original image. Each component of the resulted matrix of the tensor product ($\otimes$) is convolving with a Gaussian kernel ($K_\rho$) where $\rho > \sigma$. The eigenvectors of $J_\rho$ represent the average orientation of the gradient vector ($\hat{v}_1$) and the structure orientation ($\hat{v}_2$) at scale $\rho$.

The diffusion matrix $D(2)$ has the same eigenvectors as $J_\rho$, but its eigenvalues are chosen according to a coherence measure. This measure is proposed as the square difference between the eigenvalues of the structure tensor. The diffusion process acts mainly along the structure direction and becomes stronger as the coherence increases. In this manner, the model is even able to close interrupted lines.

Due to the characteristics of tensor $D$ (symmetry and positive eigenvalues), well posedness and scale-space properties were proved for both EED and CED models.

Based on these classical approaches, Terebes et al. [9] proposed a new model, which takes advantage of both scalar and tensor driven diffusions. The mixed-diffusion combines the CED model with an original approach to the Perona Malik filter. The model aims at using the anisotropic diffusion in case of linear structures and a scalar diffusion otherwise. In order to avoid the development of false anisotropic structures and corner rounding (caused by the CED model), the scalar diffusion is applied to the regions with a noisy background and to junctions. The decision between types of diffusion is taken with respect to the global confidence proposed by Rao [10]. A strictly tensorial approach where the amount of diffusion was weighted by a sigmoid function depending on the Rao confidence is proposed in Terebes et al [11].

Concerning the 3D applications, anisotropic diffusion has been frequently used in medical image processing. These works concern noise elimination [12] but more often boundary detection and surface extraction [13, 14]. Recently, specific PDE-based approaches were devoted to the seismic images filtering [15,16].

Furthermore, a non-PDE-based technique dedicated to the denoising of strongly oriented structures was proposed by Bakker [17]. The author combines edge preserving filtering with adaptive orientation filtering. The adaptive orientation filter consists in an elongated Gaussian filter steered by the eigenvectors of the structure tensor. Besides, a generalized Kuwahara filter, in which the window with higher confidence value is taken as a result, is proposed as edge preserving filter [18].

In this paper we present a new approach based on the CED model, dedicated to 3-D blocks processing. The aim of our method is to deliver a 3-D accurate block, while preserving the relevant discontinuities, such as edges.

In section 2 we present the general 3-D CED model and our improvements for this model. A measure will be chosen to steer the diffusion along different coherence structures, such as plane-like or line-like structures. Relevant results, on
synthetic images, will be illustrated in section 3. Finally, conclusions and further work will be presented.

2. DENOISING AND EDGE PRESERVING USING 3-D ANISOTROPIC DIFFUSION

In this section, we present the extension of CED model in the 3-D case. Thanks to a confidence measure, we propose some improvements of this filter with respect to our type of data.

2.1. 3-D CED MODEL

The 3-D model is a particular case of the general CED model [7]. The smoothed version of intensity \( U_\sigma \) is obtained after a convolution with a 3-D Gaussian kernel. The noise scale (\( \sigma \)) establishes the minimum size of the objects preserved in the smoothed image. An average of the orientation, at integration scale \( \rho \), is applied to deliver the orientation of the significant structures. Usually, the integration scale is chosen larger than the noise scale.

Due to the structure tensor properties (symmetric positive semi-definite), the eigenvalues are real and positive. These may be ordered as follows \( \mu_1 \geq \mu_2 \geq \mu_3 \).

The corresponding eigenvectors \( (v_1, v_2, v_3) \) form an orthogonal system. The largest eigenvalue carries the contrast variation in the dominant orientation of the averaged gradient vector \( (\tilde{v}_\rho) \). The orientation corresponding to the lowest contrast difference is indicated by the third vector \( (\tilde{v}_3) \).

Weickert introduces this knowledge of orientation in the general anisotropic model (2). Matrix \( D \) has the same eigenvectors as the structure tensor. The orientation of the diffusion is driven by these eigenvectors and the intensity of the process by the eigenvalues of \( D \). The author proposes the following system for choosing the eigenvalues of matrix \( D \):

\[
\begin{align*}
\lambda_1 &= \lambda_2 = \alpha \\
\lambda_3 &= \begin{cases} 
\alpha & \text{if } k = 0 \\
\alpha + (1 - \alpha) \exp\left(\frac{-C}{k}\right) & \text{otherwise}
\end{cases}
\end{align*}
\]

The parameter \( \alpha \) represents the amount of diffusivity in the orientations of the highest fluctuation contrast. In order to hamper the diffusion in these orientations, the parameter \( \alpha \) is chosen nearly 0. For theoretical reasons this parameter must be positive. The measure of coherence \( k \) is defined as:
\[ k = (\mu_1 - \mu_2)^2 + (\mu_1 - \mu_3)^2 + (\mu_2 - \mu_3)^2 \]  (5)

The threshold parameter $C$ is usually chosen equal to 1. In the coherent structures ($k \gg C$) the diffusion acts essentially along $V_3$ ($\lambda_3 \approx 1$). On the other hand, if the structure becomes isotropic ($k \to 0$), the amount of diffusivity in all three orientations is no more than $\alpha$.

This coherence measure $k$ depends on the gradient energy. For this reason, the amount of diffusivity ($\lambda_3$) in the third vector orientation always tends to 1. In conclusion, this system will smooth only in one orientation of space. This eigenvalues system is adapted to line-like structures of 3D data.

In order to deal with plane-like structures, the first idea consists in forcing the diffusion process along both the second and third eigenvectors. We can easily obtain such a result by choosing a set of eigenvalues different from (4):

\[
\begin{cases}
\lambda_1 = \alpha \\
\lambda_2 = \lambda_3 = \begin{cases}
\alpha & \text{if} \quad k = 0 \\
\alpha + (1 - \alpha) \exp \left( \frac{-C}{k} \right) & \text{otherwise}
\end{cases}
\end{cases}
\]  (6)

In the result section, the original Weickert’s approach will be denoted CED-1D and the approach based on this new set of eigenvalues will be denoted CED-2D as it allows filtering of 2D structures.

Using the CED-2D approach leads to a diffusion process steered along the 2D structures even in the presence of 1D structures. As a consequence, this method presents the drawback of smoothing the signal across 1D structures leading to a loss of relevant information. The 1D structures can be viewed as the edges of 3D or 2D structures.

Considering the behaviour of the CED-1D and CED-2D methods, we will propose a new approach which consists in choosing an appropriate set of eigenvalues to both enhancing the structures and preserving the edges as relevant details.

### 2.2 CONFIDENCE MEASURES FOR 3D DATA

We search a confident measure to discriminate between the 1D and 2D structures of data. Van Kempen et al.[19] define the notion of dimensionality of structures. In the 3-D case, beside the isotropic structures corresponding to three shift invariant orientations, two types of linear structures are possible:

– plane-like linear structure – shift invariant along two orientations,
– line-like linear structure – shift invariant along one orientation.
Analysis of the linear structure may be issued by the computation of the structure tensor. Thus, the vectors of the structure tensor point out the principal axes of orientation and the number of zero eigenvalues indicates the number of the shift invariant orientations.

A 2D surface can be viewed as plane-like structures and it is characterized by a large eigenvalue and two others close to zero. An edge is characterized by two large eigenvalues and the other close to zero. This property is due to the fact that the orientation of the average gradient around the edge is a mixture of two distinct orientations corresponding to the neighbouring regions. Thus, we can model the edge as a line-like structure.

In 2001 Bakker et al., following the works of Bigun et al. [20], proposed two measures to estimate the semblance of seismic fault with this type of linear structures:

\[ C_{\text{plane}} = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}, \quad C_{\text{line}} = \frac{\mu_2 - \mu_3}{\mu_2 + \mu_3}. \]  

These measures are combined to obtain a fault confidence. We can note that the author proposes to introduce this measure as a confidence value to select the optimal mask in an approach combining orientation adaptive filtering and edge preserving filtering. The introduction of such a priori measures leads to a technique which strongly enhances the detected faults. A serious drawback of this approach is that it leads to an important data transformation.

Considering the similarity between faults in seismic imagery and edges in three-dimensional imagery, we will propose in the next section to use this measure as an edge indicator function:

\[ C_{\text{edge}} = C_{\text{line}} (1 - C_{\text{plane}}). \]  

2.3. COHERENCE ENHANCING DIFFUSION – CED-τ D

We propose a new approach to the general CED model, more appropriate for data with strongly oriented 3D patterns. Keeping the general equation of the anisotropic diffusion (2) we introduce an adaptive system to supervise the \( D \) matrix eigenvalues.

We intend to create a system adapted to the local context, which acts in specific ways for different regions. We choose the confidence measure \( C_{\text{edge}} \) from the various set of measures dedicated to this purpose [10, 21]. We select this type of measure over other types for its close link to the nature of strongly oriented data. We propose the following system:
$\lambda_1 = \alpha,$

$\lambda_2 = \lambda_3 - (\lambda_3 - \lambda_1) h_t(\text{C}_{\text{edge}}),$ 

$\lambda_3 = \begin{cases} 
\alpha & \text{if } k = 0 \\
\alpha + (1-\alpha)\exp\left(-\frac{C}{k}\right) & \text{else},
\end{cases}$ (9)

where $h_t(s)$ is described in [11]:

$h_t(s) = \frac{\tanh[\gamma(s-\tau)]+1}{\tanh[\gamma(1-\tau)]+1},$ (10)

The eigenvalue $\lambda_2$ depends continuously on the confidence measure ($C_{\text{edge}}$) and takes values between $\lambda_1$ and $\lambda_3$. In the neighbourhood of a border, $\lambda_2$ tends to $\lambda_1$ whereas it tends to $\lambda_3$ when $C_{\text{edge}} \rightarrow 0$. Through the value of two parameters the threshold $\tau \in [0,1]$ and the slope $\gamma$, the sigmoid function $h_t(s)$ allows a better control of transition between two homogeneous regions. The function $h_t(C_{\text{edge}})$ by its argument, plays the role of a fuzzy corner and edge detector.

Within presumptive edge zones ($C_{\text{edge}} \rightarrow 1$), the process will only smooth along the smallest variation of contrast ($\nu_3$). In this case the amount of diffusivity in the first and in the second orientation given by the $\lambda_2$ and $\lambda_1$ values is equal to $\alpha$ chosen near to 0.

The regions where $C_{\text{edge}} \rightarrow 0$ are rather characterized by plane-like structures ($C_{\text{plane}} \rightarrow 1$). In these regions the process will diffuse in the plane defined by the vectors $\nu_2$ and $\nu_3$. This plane is orthogonal to the average gradient. For this type of horizons the coherence measure $k$ is high ($\mu_1 \geq \mu_2 \approx \mu_3$) and forces the $\lambda_2$ and $\lambda_3$ values to reach 1.

Because of the behaviour of our model, which can act strongly in 1D as well as 2D directions based on the $C_{\text{edge}}$ measure and the $\tau$ threshold, we call it the CED-$\tau$D diffusion.

### 3. RESULTS

This section illustrates the efficiency of our approach on synthetic blocks. The noise reduction and the edge preserving are evaluated. Our filter is compared with both the CED-1D and CED-2D models.

Since it is much easier to judge the efficiency of the algorithms on a synthetic 3D image, we propose to use a set of 15 synthetic blocks composed of oriented patterns of different frequencies and having a sinusoidal profile. Fig. 1a, 3a, 2a show respectively two front sections and a top section of a synthesized 3D block.
The 15 original-3D blocks are corrupted with additive Gaussian white noise in order to obtain 3 sets of 15 blocks corresponding to signal-to-noise-ratio (SNR) of 1 dB, 4 dB and 9 dB. Each noisy block is then filtered with our method and the CED methods. Parameters common to the various algorithms take on the same values ($dt = 0.1$, $\alpha = 10^{-4}$, $\rho = 2$, $\sigma = 1$, 24 iterations for 1dB blocks, 16 iterations for 4dB blocks and 8 iterations for 9dB blocks). In addition, parameters specific to CED-$\tau$D are set to $\tau = 0.05$ and $\gamma = 10$. We show in Figs. 1, 2 and 3 the results obtained using an explicit numerical scheme.
Firstly, we evaluate the efficiency of our method by the means of peak-signal-noise-ratio (PSNR) which allows quantifying the similarity between each diffused block and the original synthetic block:

$$\text{PSNR}(U_0, U) = 20 \log_{10} \left( \frac{\max U_0 - \min U_0}{\text{RMSE}(U_0, U)} \right).$$  \hspace{1cm} (11)$$

where $U_0$ denotes the value of the voxel with coordinates $(x,y,z)$ in the original non-noisy block, $U$ the value of the same voxel in the processed image and RMSE is the root-mean-squared-error.

<table>
<thead>
<tr>
<th>Block</th>
<th>PSNR(dB)</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>Non-Edge</td>
</tr>
<tr>
<td></td>
<td>1D</td>
</tr>
<tr>
<td>Noisy</td>
<td></td>
</tr>
</tbody>
</table>

The original block was segmented in two regions: edges and non-edges. This segmentation was achieved using a simple thresholding on the $C_{\text{edge}}$ value. Fig. 4 illustrates this segmentation step: first the $C_{\text{edge}}$ values are shown for a given slice (Fig. 4a) and then the segmentation is illustrated in Fig. 4b. For each processed
block of each set, the PSNR has been computed in these two different zones in order to illustrate the behaviour of the methods. Table 1 shows the resulting average values for each set.

Fig. 4 – a) \( C_{edge} \) block of original block 2 (Front view 29); b) binary map (edge – non-edge) obtained by imposing a threshold on \( C_{edge} \) block.

Considering the quality of the denoising, our approach performs well when compared with the CED models, in terms of both visual quality (Figs. 1, 2, 3) and global PSNR (Table 1). In particular, false anisotropic structures appear in the block processed with the CED 1D model (Figs. 1c, 2c, 3c) while our approach does not create this type of structure (Figs. 1d, 2d, 3d). This is also reflected in the PSNR values corresponding to the non-edge region. Unlike CED 2D, our model and CED 1D model preserve the edges and produce comparable PSNR values in the edge region. On the other hand CED 2D model provides a good quality in the non-edge zones (Figs. 1e, 2e, 3e).

For another global comparison between the models, we present in Table 2 the signal-to-noise-ratio (SNR) calculated for each treated and noisy block as:

\[
\text{SNR}(U, U_0) = 10 \log_{10} \frac{\sigma^2(U)}{\sigma^2(U - U_0)},
\]

(12)

where \( \sigma^2 \) is the variance. We obtained for each noisy block better SNR values than the CED classical models.

In order to investigate if the results are due to the particular choice of the test images or if they are representative for the performances of each method we performed a two-way analysis of variance [22]. The two tested factors are respectively the methods and the images. The extremely low probabilities associated to the Fisher-Snedécor test for the method effects allows us to conclude that the processing method has a significant influence over the quality of the processed result for any level of noise.
Table 2

Average SNR values for each set (1dB, 4dB, 9dB) and corresponding ANOVA

<table>
<thead>
<tr>
<th>Level of Noise</th>
<th>SNR(dB)</th>
<th>ANOVA results</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>Noisy</td>
<td>CED-1D</td>
</tr>
<tr>
<td>1dB</td>
<td>1.006</td>
<td>11.18 c</td>
</tr>
<tr>
<td>4dB</td>
<td>4.031</td>
<td>12.67 c</td>
</tr>
<tr>
<td>9dB</td>
<td>8.924</td>
<td>14.98 b</td>
</tr>
</tbody>
</table>

Any two means followed by the same letter are not significantly different at P = 0.05 within a particular level of noise (Student-Newman Keuls post-hoc test).

To compare the three methods we used the classical Student-Newman-Keuls post-hoc test [22]. We conclude that almost all the differences are significant except between CED-1D and CED-2D for the 9dB level: the result obtained using the CED-τD model are always significantly better than the CED-2D ones. The CED-1D performs generally worse than the two other methods.

Finally, considering both the noise reduction and the edge preserving, we can conclude that the proposed CED-τD model takes advantage of both the 1D and 2D Weickert’s models.

4. CONCLUSIONS

We have proposed a new approach to tensorial diffusion which takes into account the local characteristics of 3D data. More precisely, we make sure that our denoising approach preserves the edges. For this purpose we use a measure of confidence in a tensor driven diffusion process adapted to the local context. This measure allows to diffuse only in one orientation in an edge neighbourhood and to perform a diffusion process guided by two orientations along the 2D structures otherwise.

We made a qualitative and quantitative comparison of our approach with two CED models, by the means of RMSE and SNR. We show that the proposed model preserves better the edges of oriented structures and also the denoising process is more pertinent. This approach also exempts from the creation of false anisotropic structures, artefacts typically observable in images processed with the classical tensorial models. Our method can be used for preprocessing of automatic or manual interpretation of 3D data such as reflection seismic or medical data.

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