

# TWO-STEP PROCEDURE DECREASES VARIANCE IN TONE FREQUENCY ESTIMATION

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**Key words:** Frequency estimation, Real sinusoid, Pisarenko's method, Variance analysis.

We propose a two-step procedure for estimating the frequency of a single sinusoid embedded in additive, white noise. We prove its effectiveness through statistical analysis and computer experiments. Our procedure provides lower variances of the estimator than existing one-step procedures.

## 1. INTRODUCTION

In this paper, we approach the problem of the frequency estimation of a single real sinusoid from a finite number of noisy data samples. This problem is relevant for various applications, such as speech analysis, radar, sonar, communication systems, measurements, and adaptive control [1–3]. In order to solve this problem, many frequency estimation methods have been proposed and analyzed, for example the modified covariance (MC) [4, 5], the Pisarenko harmonic decomposer (PHD) [6–8], and the reformed Pisarenko harmonic decomposer (RPHD) [9–11]. For the RPHD, a closed form, asymptotically unbiased frequency estimator was introduced, based on the linear prediction (LP) property of sinusoidal signals, and on a modified least-squares (LS) cost function. The same frequency estimator has been derived in [12] from a constrained notch-filter point of view.

In Section 2, we propose a method to decrease the variance of the frequency estimator with respect to other methods, which consists of estimating an integer multiple of the frequency, and in resolving the induced aliasing by a two-step procedure. In Section 3, this procedure is illustrated using the RPHD method for the first step, and a new generalized RPHD (GRPHD) method that we introduce for the second step. Explicit relations needed for implementation and statistical

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analysis of the GRPHD are derived. Results of computer experiments, presented in Section 4, confirm the theory. Conclusions are drawn in the last section.

## 2. TWO-STEP PROCEDURE

In many single sinusoid frequency estimation techniques, the estimate  $\hat{\omega}_{01}$  of the unknown frequency  $\omega_0$  is calculated from an estimator  $\rho_1^* = \cos(\hat{\omega}_{01})$  using  $\hat{\omega}_{01} = \cos^{-1}(\rho_1^*)$  [11]. If the estimator is unbiased, the relation between the variances of  $\hat{\omega}_{01}$  and  $\rho_1^*$  is  $\text{var}(\hat{\omega}_{01}) \approx \frac{\text{var}(\rho_1^*)}{\sin^2(\omega_0)}$ . We propose to generalize this procedure by finding an estimator  $\rho_k^* = \cos(\hat{\omega}_{0k})$  of an integer multiple  $k\omega_0$  of the frequency  $\omega_0$ . The roots of this equation are

$$\hat{\omega}_{0j} = \frac{1}{k} \left( (-1)^{j-1} \cos^{-1}(\rho_k^*) + \left[ \frac{j}{2} \right] 2\pi \right), \quad j = 1, 2, \dots, k, \quad (1)$$

where we have denoted by  $[x]$  the largest integer smaller than  $x$ . If we dispose of a first, rough frequency estimate  $\hat{\omega}_{01}$ , then the second frequency estimate corresponds to  $j = 1 + \left\lfloor \frac{k\hat{\omega}_{01}}{\pi} \right\rfloor$ .

As we will show below by statistical analysis and computer experiments for the GRPHD, the variance of the second estimate is smaller than the variance of the first one. An a priori argument might be as follows.

If the estimator  $\rho_k^*$  is unbiased, its variance can be evaluated from:

$$\begin{aligned} \text{var}(\rho_k^*) &= E \left\{ \left( \cos(k\omega_0) - \cos(\hat{\omega}_{0k}) \right)^2 \right\} = E \left\{ \left( \cos(k\omega_0) - \cos(k\hat{\omega}_0) \right)^2 \right\} = \\ &= E \left\{ 4 \sin^2 \left( \frac{k(\hat{\omega}_0 + \omega_0)}{2} \right) \sin^2 \left( \frac{k(\hat{\omega}_0 - \omega_0)}{2} \right) \right\}. \end{aligned} \quad (2)$$

For small  $k$  and for a good estimation (small  $|\hat{\omega}_0 - \omega_0|$ ), we have

$$\text{var}(\hat{\omega}_0) \approx \frac{\text{var}(\rho_k^*)}{k^2 \sin^2(k\omega_0)}. \quad (3)$$

In frequency estimation methods such as PHD and RPHD, the variances of the frequency estimates are evaluated in terms of some quantities proportional to the autocorrelation function of the signal, calculated at lags 0, 1, and 2. As we will show later, the variance (3) can also be evaluated in terms of quantities proportional to the signal's autocorrelation function, calculated at lags 0,  $k$ , and  $2k$ . Thus, for a sufficiently large number of data points  $N$ , and comparatively small  $k$ ,  $\text{var}(\rho_k^*)$  increases much less with respect to  $\text{var}(\rho_1^*)$  than the increase of the denominator in (3) with  $k$ .

The variance in (3) is unbounded at frequencies  $\omega_{Mi} = i \frac{\pi}{k}$ ,  $i = 0, 1, \dots, k$ , and minimum at

$$\omega_{mi} = \left(i + \frac{1}{2}\right) \frac{\pi}{k}, \quad i = 0, 1, \dots, k-1. \quad (4)$$

In the two-step procedure, we use  $N$  data points and a method such as MC, PHD or RPHD in order to obtain a first estimate  $\hat{\omega}_{01}$ , and to choose a value of  $k \ll N$  such that the first estimate be as close as possible of an  $\omega_{mi}$ . We then refine the search by using the same  $N$  data points in order to estimate the  $\rho_k^* = \cos(\hat{\omega}_{0k})$ .

### 3. TWO-STEP GRPHD METHOD

We analyzed and tested the two-step procedure by using the RPHD method for the first step, and the GRPHD, to be introduced below, for the second step.

In single frequency estimation, the following signal model is considered [11]:

$$x(n) = s(n) + q(n) = \alpha \cos(\omega_0 n + \varphi) + q(n), \quad n = 1, 2, \dots, N, \quad (5)$$

where  $\alpha > 0$ ,  $\omega_0 \in (0, \pi)$ , and  $\varphi$  are the unknown amplitude, frequency and phase of the sinusoid, and  $q(n)$  is assumed to be a Gaussian white noise with zero mean and variance  $\sigma_q^2$ . We denote the signal-to-noise ratio  $\text{SNR} = \alpha^2 / (2\sigma_q^2)$ . The sinusoid is linearly predictable from the past samples according to:

$$s(n) = 2 \cos(k\omega_0) s(n-k) - s(n-2k), \quad k = 1, 2, \dots \quad (6)$$

a fact that suggests the definition of an error function  $e_k(n) = x(n) - 2 \cos(k\lambda) x(n-k) + x(n-2k)$ , where  $\lambda$  is the parameter to be determined.

However, by generalizing the RPHD method introduced in [11], in order to obtain an asymptotically unbiased frequency estimator we must define a modified error function  $\varepsilon_k(n) = \frac{e_k(n)}{\sqrt{2(2 + \cos(2k\lambda))}}$ , and the corresponding LS cost function

$$J_{kN}(\lambda) = \sum_{n=2k+1}^N \varepsilon_k^2(n). \text{ Solving } \frac{dJ_{kN}(\lambda)}{d\lambda} = 0 \text{ yields}$$

$$2A_{kN} \cos^2(k\lambda) - B_{kN} \cos(k\lambda) - A_{kN} = 0, \quad (7)$$

where

$$A_{kN} = \sum_{n=2k+1}^N (x(n) + x(n-2k))x(n-k), \quad (8)$$

$$B_{kN} = \sum_{n=2k+1}^N ((x(n) + x(n-2k))^2 - 2x^2(n-k)). \quad (9)$$

The root of (7) that provides the estimate is

$$\rho_k^* = \frac{B_{kN} + \sqrt{B_{kN}^2 + 8A_{kN}^2}}{4A_{kN}}. \quad (10)$$

The above equations generalize the corresponding results from [11], which can be obtained by setting  $k = 1$ .

For a theoretical evaluation of the proposed two-step procedure, we calculated the variance of the new estimate by generalizing the analysis followed in [11]. The variance analysis technique is based on the definition of a second order polynomial motivated by (7)

$$f(\rho) = 2A_{kN}\rho^2 - B_{kN}\rho - A_{kN} \quad (11)$$

on (3), and on the following formula [11]

$$\text{var}\{\rho_k^*\} \approx \frac{E\{f^2(\rho)\}}{(E\{f'(\rho)\})^2} \bigg|_{\rho=\cos(k\omega_0)}. \quad (12)$$

By using  $\rho = \cos(k\omega_0)$  and (11) we obtain

$$\begin{aligned} f^2(\rho) \big|_{\rho=\cos(k\omega_0)} &= \cos^2(2k\omega_0) A_{kN}^2 - 2\cos(k\omega_0)\cos(2k\omega_0) A_{kN} B_{kN} + \\ &+ \cos^2(k\omega_0) B_{kN}^2, \end{aligned} \quad (13)$$

$$f'(\rho)\big|_{\rho=\cos(k\omega_0)} = 4\cos(k\omega_0)A_{kN} - B_{kN}. \quad (14)$$

In order to compute the variance of  $\hat{\omega}_0$  using (3) and (12), the values of  $E\{A_{kN}\}$ ,  $E\{B_{kN}\}$ ,  $E\{A_{kN}^2\}$ ,  $E\{B_{kN}^2\}$ , and  $E\{A_{kN}B_{kN}\}$  are required. The main steps we followed in computing these terms are outlined in the Appendix, together with an argument for the unbiasedness of the estimator. We obtained:

$$\begin{aligned} \text{var}\{\hat{\omega}_0\} \approx & \frac{2k + (2 + \cos(2k\omega_0))\beta(k+1, N-k) - 2\beta(k+1, N-2k)}{2k^2 \text{SNR} (N - 2k + \beta(k+1, N-k))^2 \sin^2(k\omega_0)} - \\ & - \frac{2\beta(2k+1, N-k) - \beta(2k+1, N-2k)}{2k^2 \text{SNR} (N - 2k + \beta(k+1, N-k))^2 \sin^2(k\omega_0)} + \\ & + \frac{(2N-5k)\cos^2(2k\omega_0) + (2N-4k)\cos^2(k\omega_0)}{2k^2 \text{SNR}^2 (N - 2k + \beta(k+1, N-k))^2 (2 + \cos(2k\omega_0))^2 \sin^2(k\omega_0)}, \end{aligned} \quad (15)$$

where

$$\begin{aligned} \beta(k_1, k_2) &= \sum_{n=k_1}^{k_2} \cos(2(\omega_0 n + \varphi)) \\ &= \frac{\sin(\omega_0(k_2 - k_1 + 1))\cos(\omega_0(k_2 + k_1) + 2\varphi)}{\sin(\omega_0)}. \end{aligned} \quad (16)$$

An asymptotic form of this variance results by formally making  $\beta = 0$  in (15). Due to the closed-form, simple structure of (15), the asymptotic form brings no major simplifications, but the dependence of the variance on  $\varphi$  is lost. For  $k=1$ , this asymptotic variance reduces to the corresponding one calculated in [11], except for minor differences probably caused by unimportant typing errors in the original work.

The decrease in the calculated variances for the two-step method as compared to the one-step method is illustrated in Fig. 1, where we have plotted (15) in function of frequency, for  $k=2$ , and 3, at  $N=20$ ,  $\text{SNR}=10$  dB, and  $\varphi=0$ . We have also shown (15) for  $k=1$  (one-step method). It is clear that, for every frequency, there exist values of  $k>1$  such that the variance provided by the two-step method is lower than the variance provided by the one-step method. The

smallest errors correspond indeed to frequencies around those defined in (4). Similar results are obtained with the asymptotic form of (15).

#### 4. RESULTS OF COMPUTER EXPERIMENTS

Computer experiments have been carried out in order to evaluate the proposed two-step method by comparing it with the RPHD method [11].

The data sequences for single real, noisy sinusoids were generated using (5), with  $\alpha = \sqrt{2}$  and  $\varphi = 0$ . Different noise variances  $\sigma_q^2$  were chosen in order to obtain the desired SNRs.

The first step in our method was identical with the RPHD method and gave an initial frequency estimate  $\hat{\omega}_{01}$  that corresponded to  $k = 1$  (although any other estimation method could be used in this first step). The purpose of computer experiments was to prove that one can choose some  $k > 1$  for the second step, in order to obtain a better frequency estimate.

We made use of (8–10), and (1) with the appropriate value for  $j$ , with  $k = 2$  and  $3$ , for finding the second-step GRPHD estimates. In order to evaluate the frequency estimator performances, the mean square frequency errors (MSFEs) were computed by averaging 500 independent runs. These measured MSFEs for  $k = 1, 2$  and  $3$  are represented in every figure that illustrates the experimental results, together with the corresponding variances (15).

In Fig. 1, the MSFEs are plotted versus  $\omega_0$ , for  $N = 20$ , SNR = 10 dB, and  $\varphi = 0$ . Note that, for every frequency, one can choose  $k = 2$  or  $k = 3$  in order to get a better frequency estimate than in the case  $k = 1$ . Similar results were obtained for other values of  $\varphi$ .

In Fig. 2 the MSFEs versus SNR are represented, at  $\omega_0 = 0.1\pi$ ,  $N = 20$ , and  $\varphi = 0$ . The errors are smaller with the two-step method as compared with the one-step method for all values of the SNR. Note the excellent agreement between the calculated and experimental variances, except for very low values of the SNR.

Figure 3 shows the MSFEs in function of  $N$  for  $\omega_0 = 0.1\pi$ , SNR = 10 dB, and  $\varphi = 0$ . It can be noticed the decrease of the variance and of the MSFE for all values of  $N$ . Experiments have been performed for various values of  $\varphi$ , with similar results.

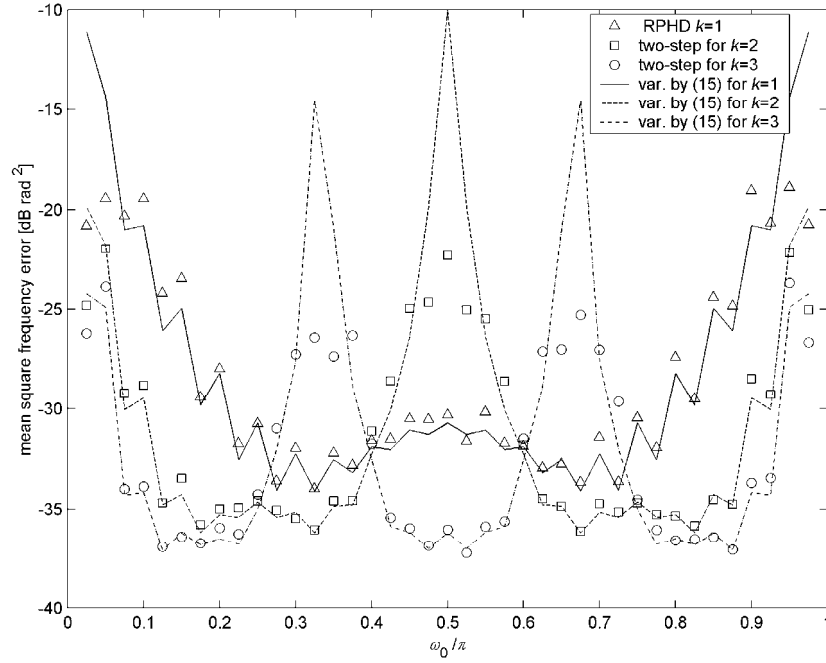


Fig. 1 – Mean square frequency errors and variances *versus*  $\omega_0$  at SNR = 10 dB,  $N = 20$ , and  $\varphi = 0$ .

## 5. CONCLUSIONS

We proposed to decrease the errors in the estimation of the frequency of a sinusoid embedded in additive, gaussian white noise by means of a two-step procedure. In the first step, we applied a known one-step method in order to find a rough estimate of the frequency. We proposed a new GRPHD method for second step in order to estimate an integer multiple of the frequency, and used the first estimate for solving the induced aliasing.

We calculated the variance of our estimate and its asymptotic form, and showed they gave better results than the corresponding ones provided by the one-step procedure. We also proved that our estimate was asymptotically unbiased.

We performed several computer experiments in order to prove that our results were correct, and to illustrate the decrease of the MSFE in function of various parameters of the problem.

We can conclude that the addition of a second step to the frequency estimation of a sinusoid significantly improves the variance of the estimator.

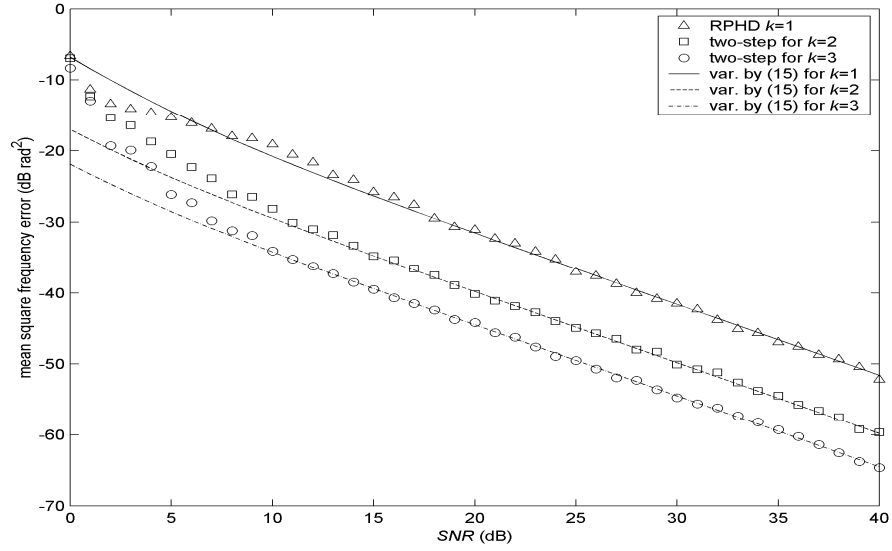


Fig. 2 – Mean square frequency errors and variances *versus* SNR at  $\omega_0 = 0.1\pi$ ,  $N = 20$ , and  $\varphi = 0$ .

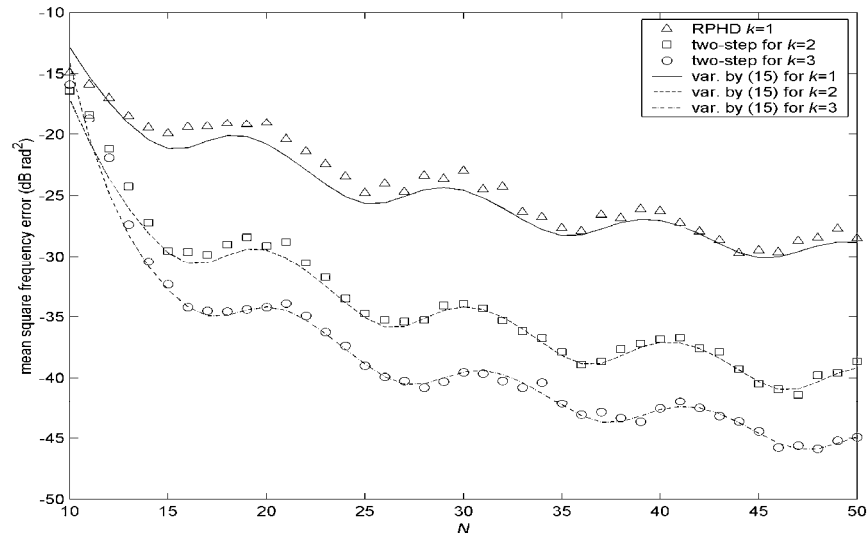


Fig. 3 – Mean square frequency errors and variances *versus*  $N$  at  $\omega_0 = 0.1\pi$ , SNR = 10 dB, and  $\varphi = 0$ .

## APPENDIX

The quantities  $A_{kN}$  and  $B_{kN}$  can be written as sums of three uncorrelated terms. For example, we can use (5), (6), (8) and (16) in order to get  $A_{kN} = T_{A1} + T_{A2} + T_{A3}$ , with

$$\begin{aligned} T_{A1} &= \sum_{n=2k+1}^N 2\alpha^2 \cos(k\omega_0) \cos^2(\omega_0(n-k) + \varphi) \\ &= \alpha^2 (N - 2k + \beta(k+1, N-k)) \cos(k\omega_0), \end{aligned} \quad (\text{A1})$$

$$T_{A2} = \sum_{n=2k+1}^N \alpha \cos(\omega_0(n-k) + \varphi) \times (2 \cos(k\omega_0) q(n-k) + q(n) + q(n-2k)), \quad (\text{A2})$$

$$T_{A3} = \sum_{n=2k+1}^N q(n-k) (q(n) + q(n-2k)). \quad (\text{A3})$$

We have  $E\{T_{A2}\} = E\{T_{A3}\} = 0$ , and  $E\{A_{kN}\} = E\{T_{A1}\} = T_{A1}$ .

In order to calculate  $E\{A_{kN}^2\}$ , we note that  $E\{T_{Ai}T_{Aj}\} = 0$  for  $i \neq j$ . Therefore

$$E\{A_{kN}^2\} = T_{A1}^2 + E\{T_{A2}^2\} + E\{T_{A3}^2\}. \quad (\text{A4})$$

The second term in the RHS of (A4) can be evaluated as follows

$$\begin{aligned} E\{T_{A2}^2\} &= \sum_{n=2k+1}^N \alpha^2 \cos^2(\omega_0(n-k) + \varphi) \times \left[ 4 \cos^2(k\omega_0) E\{(q(n-k))^2\} + \right. \\ &\quad \left. + E\{(q(n))^2\} + E\{(q(n-2k))^2\} \right] + \\ &\quad + 2 \sum_{n=2k+1}^N \sum_{m=n+1}^N \alpha^2 \cos(\omega_0(n-k) + \varphi) \cos(\omega_0(m-k) + \varphi) \times \\ &\quad \times E\{(2 \cos(\omega_0 k) q(n-k) + q(n) + q(n-2k)) \times \\ &\quad \times (2 \cos(\omega_0 k) q(m-k) + q(m) + q(m-2k))\} = \\ &= \sum_{n=2k+1}^N \alpha^2 \frac{1 + \cos(2\omega_0(n-k) + 2\varphi)}{2} (4 \cos^2(k\omega_0) \sigma_q^2 + 2 \sigma_q^2) + \\ &\quad + \sum_{n=2k+1}^N \sum_{m=n+1}^N \alpha^2 [\cos(\omega_0(n+m-2k) + 2\varphi) + \cos(\omega_0(m-n))] \times \\ &\quad \times [2 \cos(\omega_0 k) E\{q(n-k)q(m-2k)\} + 2 \cos(\omega_0 k) E\{q(n)q(m-k)\} + \\ &\quad + E\{q(n)q(m-2k)\}]. \end{aligned}$$

Hence:

$$\begin{aligned} E\{T_{A2}^2\} &= \alpha^2 \sigma_q^2 [4N - 10k + (4N - 12k) \cos(2k\omega_0) + \\ &+ \beta(k+1, N-k)(2 + \cos(2k\omega_0)) + 2\beta(k+1, N-2k) + \\ &+ 2\beta(2k+1, N-k) + \beta(2k+1, N-2k)] . \end{aligned} \quad (A5)$$

With similar calculations, the expression of the last term in the RHS of (A5) can be shown to be  $E\{T_{A3}^2\} = (4N - 10k) \sigma_q^4$ , so that

$$\begin{aligned} E\{A_{kN}^2\} &= \alpha^4 (N - 2k + \beta(k+1, N-k))^2 \cos^2(k\omega_0) + \\ &+ \alpha^2 \sigma_q^2 [4N - 10k + (4N - 12k) \cos(2k\omega_0) + \\ &+ \beta(k+1, N-k)(2 + \cos(2k\omega_0)) + 2\beta(k+1, N-2k) + 2\beta(2k+1, N-k) + \\ &+ \beta(2k+1, N-2k)] + \sigma_q^4 (4N - 10k). \end{aligned} \quad (A6)$$

The steps we followed in order to calculate  $E\{B_{kN}\}$  and  $E\{B_{kN}^2\}$  are similar to those presented above, and will be skipped for brevity. The results are

$$E\{B_{kN}\} = \alpha^2 (N - 2k + \beta(k+1, N-k)) \cos(2k\omega_0), \quad (A7)$$

$$\begin{aligned} E\{B_{kN}^2\} &= \alpha^4 (N - 2k + \beta(k+1, N-k))^2 \cos^2(2k\omega_0) + \\ &+ \alpha^2 \sigma_q^2 [4N + 4(N - 4k) \cos(4k\omega_0) + 8\beta(k+1, N-k)(2 + \cos(2k\omega_0)) \\ &- 16\beta(k+1, N-2k) - 16\beta(2k+1, N-k) + \\ &+ 8\beta(2k+1, N-2k)(1 + \cos(2k\omega_0))] + \sigma_q^4 (4N - 8k). \end{aligned} \quad (A8)$$

Based on the decomposition of  $A_{kN}$  and  $B_{kN}$  in sums of three uncorrelated terms, it can be shown that  $E\{A_{kN} B_{kN}\} = T_{C1} + T_{C2} + T_{C3}$ , with

$$T_{C1} = \alpha^4 \cos(k\omega_0) \cos(2k\omega_0) (N - 2k + \beta(k+1, N-k))^2, \quad (A9)$$

$$\begin{aligned} T_{C2} &= \alpha^2 \sum_{n=2k+1}^N \sum_{m=2k+1}^N 4 \cos(\omega_0(n-k) + \varphi) \cos(\omega_0(m-k) + \varphi) \times \\ &\times E\{(2 \cos(k\omega_0) q(n-k) + q(n) + q(n-2k)) \times \\ &\times (\cos(k\omega_0)(q(m) + q(m-2k) - q(m-k)))\}, \end{aligned} \quad (A10)$$

$$T_{C3} = E \left\{ \sum_{n=2k+1}^N \sum_{m=2k+1}^N q(n-k)(q(n) + q(n-2k)) \times \right. \\ \left. \times (q^2(m) + q^2(m-2k) + 2q(m)q(m-2k) - 2q^2(n-k)) \right\} = 0. \quad (A11)$$

After performing the calculations we obtained

$$E \{ A_{kN} B_{kN} \} = \alpha^4 (N - 2k + \beta(k+1, N-k))^2 \cos(k\omega_0) \cos(2k\omega_0) + \\ + \alpha^2 \sigma_q^2 \left[ (4N - 14k)(\cos(k\omega_0) + \cos(3k\omega_0)) + 2(\beta(k+1, N-2k) + \right. \\ \left. + \beta(2k+1, N-k)) \frac{\cos(2k\omega_0)}{\cos(k\omega_0)} + 4\beta(2k+1, N-2k) \cos(k\omega_0) \right]. \quad (A12)$$

The proof that  $\rho_k^* = \cos(\hat{\omega}_{0k})$  is asymptotically unbiased is a generalized version of the proof introduced in [11] for the case  $k=1$ . Namely, a first order Taylor series evaluation of  $f$  at  $\rho_k^*$  starting from  $f$  at  $\rho_k = \cos(k\omega_0)$  gives  $f(\rho_k^*) = 0 = f(\cos(k\omega_0)) + (\rho_k^* - \cos(k\omega_0))f'(\cos(k\omega_0))$ , wherefrom:

$$\cos(\hat{\omega}_{0k}) = \cos(k\omega_0) - \frac{f(\rho)}{f'(\rho)} \Big|_{\rho=\cos(k\omega_0)}. \quad \text{Therefore it must be proved that}$$

$$\lim_{N \rightarrow \infty} \frac{f(\rho)}{f'(\rho)} \Big|_{\rho=\cos(k\omega_0)} = 0. \quad \text{The proof is based on the ergodic and statistical properties}$$

of the signal and noise, and the main steps can be outlined as follows:

$$\lim_{N \rightarrow \infty} \frac{f(\rho)}{f'(\rho)} \Big|_{\rho=\cos(k\omega_0)} = \lim_{N \rightarrow \infty} \frac{A_{kN} \cos(2k\omega_0) - B_{kN} \cos(k\omega_0)}{4A_{kN} \cos(k\omega_0) - B_{kN}} = \\ = \frac{0}{\alpha^2 (4\cos^2(k\omega_0) - \cos(2k\omega_0)) + 0} = 0.$$

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