ITERATIVE PROCEDURE FOR REAL SINGLE-TONE FREQUENCY ESTIMATION

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We propose a simple, iterative procedure for the frequency estimation of a real sinusoid corrupted by additive, white noise, based on a combination of a known low-complexity method and filtering. We obtain mean square errors close to the Cramer-Rao lower bound, for relatively low signal-to-noise ratios in a few iterations, with a reasonable increase in computation complexity.

1. **INTRODUCTION**

In many situations, a signal consisting of a single tone corrupted by additive white noise is available, and the frequency of the tone must be estimated. This problem has applications in communications, radar, sonar, measurements, adaptive control, speech processing etc [1], and it belongs to the wider class of spectrum estimation problems [2]. Both complex and real signals have been considered in the literature. As maximum likelihood (ML) estimators perform very well but are not computationally efficient [3, 4], various estimation algorithms have been proposed (see [5−11] for the complex case and [4, 12, 13, 14, 15] for the real case).

The Pisarenko harmonic decomposer (PHD) [12, 13, 16, 17,], and the reformed Pisarenko harmonic decomposer (RPHD) [15, 18, 19] are among the algorithms that perform well and are computationally efficient in the real case, when the signal consists of a real sinusoid and an additive, white noise. The theory for both the above mentioned algorithms assumes a white noise. In the present paper we report an iterative method designed to improve the performance of the RPHD such that frequency error variances close to the Cramer-Rao lower bound (CRLB) are obtained for signal-to-noise ratios (SNR) as low as 3 dB in a few iterations. An iteration consists of filtering the initial data sequence with a frequency selective filter whose maximum of the frequency response is at the

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frequency estimate obtained in the previous iteration and in computing a better frequency estimate based on the data sequence at the filter output by means of the RPHD. Although the noise in the filtered sequence is no more white, experiments show that the considered estimators perform well in this case too.

We present the algorithm in Section 2, together with an analysis of the increase in the SNR due to filtering. In Section 3 we report results of computer experiments, while conclusions are drawn in Section 4.

2. ITERATIVE PROCEDURE

We consider the following signal model:

$$x(n) = s(n) + q(n) = \alpha\cos(\omega_0 n + \varphi) + q(n),\ n = 1,2,\ldots,N \tag{1}$$

where $\alpha > 0$, $\omega_0 \in (0,\pi)$ and φ are the deterministic but unknown amplitude, (angular) frequency and phase of the sinusoid respectively, and $q(n)$ is a zero-mean Gaussian white noise of variance σ^2.

The signal-to-noise ratio is

$$\mathrm{SNR}_0 = \frac{\alpha^2}{2\sigma^2}. \tag{2}$$

In order to estimate the frequency we propose the following procedure: we initialize the algorithm by using the RPHD for the calculation of an initial estimate $\hat{\omega}_0$ for ω_0; then we start iterations such that, at iteration $k \geq 1$, we filter the initial data sequence with a selective filter centered at the frequency estimated at iteration k–1, $\hat{\omega}_{k-1}$, and we apply to the signal from the filter output the RPHD in order to calculate a new frequency estimate $\hat{\omega}_k$.

The purpose of filtering the data sequence is to increase the SNR. The filtered signal contains a colored noise component that does not fit into the theory of the frequency estimation algorithm we have mentioned, so that its performance has to be tested for this case.

We have considered the second-order noise rejection filter with the following transfer function:

$$H(z) = \frac{z^2}{z^2 - 2\rho\cos(\omega_r)z + \rho^2}, \tag{3}$$

where ρ is a parameter close to unity in order to provide selectivity (but smaller, for stability) and, at each iteration k of the algorithm we have made $\omega_r = \hat{\omega}_{k-1}$.

The performance of the noise-rejection filter can be illustrated by evaluating the enhancement η of the SNR

$$\eta = \frac{\mathrm{SNR}_1}{\mathrm{SNR}_0}, \tag{4}$$

where $\mathrm{SNR}_1 = P_{1s}/P_{1q}$ is defined at the filter output and SNR_0 has been defined in (2) (we have denoted by P_{1s} and P_{1q} the signal power and noise power at the filter output respectively). For evaluation purposes, we consider $\omega_r = \omega_0$.

Denoting further by $P_s(\omega) = \frac{\pi A^2}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ and $P_q(\omega) = \sigma^2$ the power spectral densities of the signal and noise defined in (1) respectively, and by taking into account their statistical independence, we have

$$P_{1s} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H(\mathrm{e}^{\mathrm{j}\omega})\right|^2 P_x(\omega)\,\mathrm{d}\,\omega = \frac{\alpha^2}{2(1-\rho)^2\left((1+\rho)^2 - 4\rho\cos^2\omega_0\right)} \tag{5}$$

$$P_{1q} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H(\mathrm{e}^{\mathrm{j}\omega})\right|^2 P_q(\omega)\,\mathrm{d}\,\omega = \frac{\sigma^2(1+\rho^2)}{(1-\rho^2)\left(\rho^4 - 2\rho^2\cos(2\omega_0) + 1\right)} \tag{6}$$

The integral in (5) is a simple application of the properties of the Dirac pulse, while the integral in (6) can be calculated in a straightforward way by means of the residuum theory. Simpler approximate expression are obtained by taking into account that $\rho < 1, \rho \cong 1$. The above results become:

$$P_{1s} \cong \frac{\alpha^2}{8(1-\rho)^2\sin^2\omega_0}, \tag{7}$$

$$P_{1q} \cong \frac{\sigma^2}{4(1-\rho)\sin^2\omega_0}. \tag{8}$$

Finally, using (4), (7) and (8) we get

$$\eta = \frac{1}{1-\rho}. \tag{9}$$

Equation (9) reveals an appreciable increase of the signal-to-noise ratio if we take into account the range of ρ $\left(\rho < 1, \rho \cong 1\right)$. These noise rejection properties of the filter and the fact that, due to the good estimation provided by the RPHD, the estimated frequency does not differ significantly from the true frequency are in support of an iterative procedure that consists of computing $\hat{\omega}_k$, $k = 1, 2,\ldots$ by using the signal at the filter output, with the previous estimate $\hat{\omega}_{k-1}$ substituted for the parameter ω_r in the expression of the frequency response. The variance of the estimator decreases at each iteration as the estimated frequency approaches gradually the true frequency. These a priori arguments are confirmed by experimental results presented in the next section.

3. **SIMULATION RESULTS**

Computer experiments have been performed in order to evaluate the proposed iterative procedure. For frequency estimation, at each step of the algorithm we used the RPHD method [15]. The data sequence for single real, noisy sinusoids was generated using (1) with $\alpha = 2$ and $\varphi = 0$. We compared the estimates resulted from the iterative procedure after one iteration ($k = 1$) and $k = 5$ iterations, to the RPHD method (corresponding to the initialization step of the algorithm) and to the CRLB [20]. The parameter ρ was chosen experimentally. The simulation results presented below are averages of 1000 independent runs.

The results illustrated in Figs. 1, 2 and 3 indicate the decrease of the mean square frequency errors with the number of iterations. After five iterations, the estimator performance approaches the CRLB.

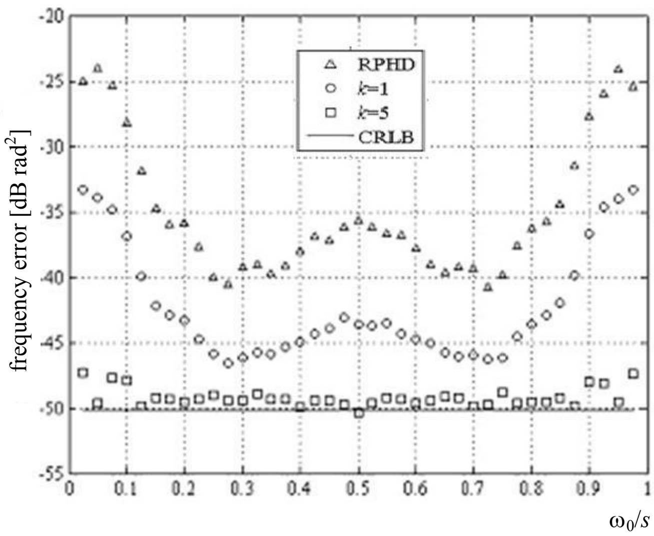

Fig. 1 – Mean square frequency errors *versus* frequency for $N = 50$, SNR = 10 dB, $\rho = 0.98$.

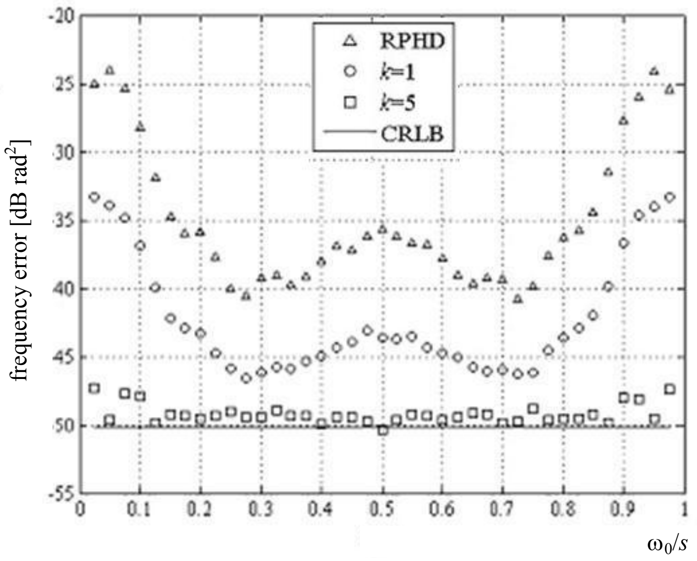

Fig. 2 – Mean square frequency *errors* versus frequency for $N = 100$, SNR = 3 dB, $\rho = 0.95$.

In experiments corresponding to Figs. 1 and 2, relatively low signal-to-noise ratios were chosen in order to prove the efficiency of the noise rejection filter. The larger SNR for the situation in Fig. 1 with respect to the one in Fig. 2 allowed for a smaller number of signal samples. Generally, a larger value of the SNR allows for a smaller number of signal samples and a value of ρ closer to 1 for achieving good estimates in a low number of iterations. Note that the frequency dependency of the mean square error, which occurs in real sinusoid frequency estimation, tends to decrease with the number of iterations. Experiment illustrated in Fig. 3 show that, at large SNR's, the estimator performances approaches the CRLB after one iteration.

The simulation results indicate that the desired performance of the algorithm is achieved in a few iterations. Consequently, the additional computation complexity introduced by iterations does not change the order of magnitude of the complexity of the original estimation method. In our examples, the RPHD (used at the initialization of the algorithm and then at each iteration) takes $3N$–4 real multiplications, $4N$–8 real additions and 5 other operation that are usually implemented by ROM accesses. Additionally, each iteration involves $2N$–2 real multiplications and $2N$–3 real additions for filtering and an RPHD. Therefore, for k iterations, the algorithm takes $(5k+3)N-6k-4$ real multiplications, $(6K+4)N-11k-8$ real additions and $5k+5$ other operations. As shown above, a value of $k = 5$ is sufficient for achieving a good performance.

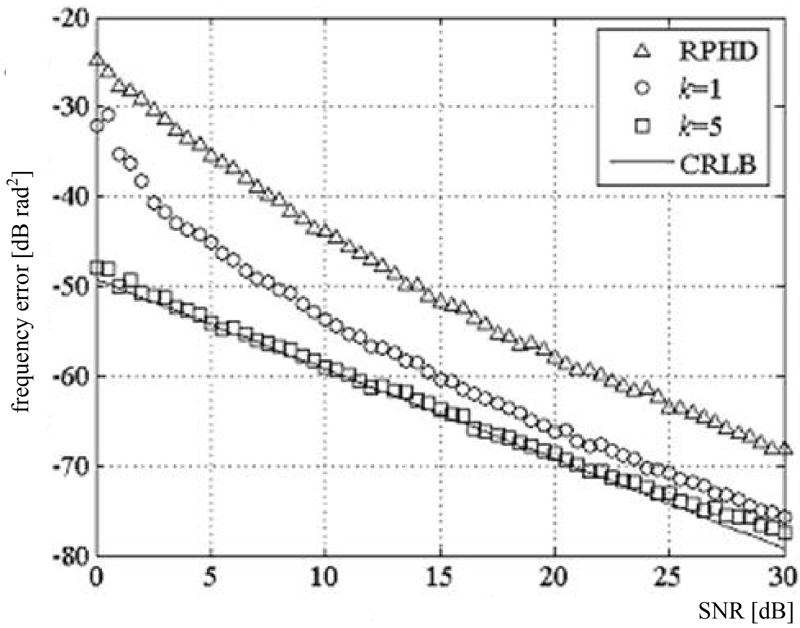

Fig. 3 – Mean square frequency errors *versus* SNR, for $f = 0.3$, $n = 100$, $\rho = 0.98$.

4. CONCLUSIONS

We have proposed an iterative algorithm for the estimation of the frequency of a signal consisting of a real sinusoid and an additive white noise. Starting from a known, single-step method (RPHD), we have used that method as a first step of the algorithm, then filtered the initial data sequence with a frequency selective filter centered on that estimate and used again that method on the filtered data in order to obtain the next estimate. This process has then been iterated.

Filtering has been introduced in order to significantly improve the signal-to-noise ratio. Although after filtering the noise gets colored, experimental results show that the RPHD still provides good frequency estimates.

Experiments indicate that this iterated procedure leads to an estimated frequency with a mean square error close to the CRLB, significantly improving the initial, first-step estimate in just a few iterations (below ten, tipically five). As the number of iterations is low, the asymptotic complexity of the algorithm is the same as in the initial method.

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