CHOICE OF THE WINDOW USED IN THE INTERPOLATED DISCRETE FOURIER TRANSFORM METHOD

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Key words: Interpolated Discrete Fourier Transformation (IpDFT) method, Sine-wave parameter estimation, Maximum sidelobe decay windows.

In this paper a criterion for optimal choice of the maximum sidelobe decay window used in the interpolated discrete Fourier transform (IpDFT) method for estimating the parameters of a sine-wave is proposed. The window chosen by the proposed criterion is the window with the lowest order, which ensures that the systematic errors that affect the sine-wave parameters estimates are very small compared with the quantization errors. The window is chosen as a function of the maximum sine-wave amplitude up to which the systematic errors that affect the frequency estimates are very small compared with the quantization errors. To confirm the effectiveness of the proposed criterion, some computer simulations and experimental results are presented.

1. INTRODUCTION

In many engineering applications it is very important to know with high accuracy the parameters of a sine-wave. The methods used for this purpose can be classified in time-domain methods (parametric methods) and frequency-domain methods (non-parametric methods) [1]. Due to the availability of Fourier analysis packages and of the robustness towards the signal model inaccuracies the frequency-domain methods are frequently used.

A frequency-domain method often used for estimating the parameters of a sine-wave in non-coherent sampling mode is the interpolated discrete Fourier transform (IpDFT) method [2–6]. The best performances are obtained using the maximum sidelobe decay windows (or class I Rife-Vincent windows) since in this case the parameters can be estimated by analytical formulas [4, 6]. Unfortunately, in the scientific literature no criterion is given for the optimal choice of the windows used in the IpDFT method for estimating the sine-wave parameters.

This paper is focused on the determination of a criterion for optimal choice of the maximum sidelobe decay window used in the IpDFT method for estimating the

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Rev. Roum. Sci. Techn. – Électrotechn. et Énerg., **54**, *4*, pp. 365–374, Bucarest, 2009

parameters of a sine-wave. The effectiveness of the proposed criterion is verified by means of computer simulations and experimental results as well.

2. **THEORETICAL BACKGROUND**

Let us consider a sine-wave sampled at a known frequency, f_s

$$x(m) = A\sin\left(2\pi\frac{f_{in}}{f_s}m + \varphi\right), \quad m = 0,1\ldots,M-1, \tag{1}$$

where A, f_{in} and φ are respectively the amplitude, frequency, and phase of the sine-wave and M is the number of acquired samples. In order to satisfy the Nyquist criterion, f_{in} is smaller than $f_s/2$.

The relationship between the frequencies f_{in} and f_s is given by

$$\frac{f_{in}}{f_s} = \frac{\lambda_0}{M} = \frac{l+\delta}{M}, \tag{2}$$

in which l and δ are respectively the integer and the fractional parts of the number of the acquired sine-wave cycles λ_0. δ is related to the non-coherent sampling mode and $-0.5 \le \delta < 0.5$.

For $\delta = 0$ the sampling process is coherent with the input sine-wave (coherent sampling mode) and (2) represents the coherent sampling relationship between frequencies f_{in} and f_s.

For $\delta \neq 0$ the sampling process is non-coherent with the input sine-wave (non-coherent sampling mode) and the coherent sampling relationship is not fulfilled. In this situation the well-known windowing approach is used, leading to the analysis of the signal $x_w(m) = x(m)\cdot w(m)$, where $w(\cdot)$ is the window used. The discrete-time Fourier transform (DTFT) of the signal $x_w(\cdot)$ is given by

$$X_w(\lambda) = \frac{A}{2j}\left[W(\lambda-\lambda_0)e^{j\varphi} - W(\lambda+\lambda_0)e^{-j\varphi}\right],\ \lambda \in [0,\ M), \tag{3}$$

where λ represents the continuous frequency expressed in bins and $W(\cdot)$ is the DTFT of $w(\cdot)$.

The second term in (3) represents the image part of the sine-wave spectrum.

The parameters of a sine-wave can be very accurately estimated by the IpDFT method. Its best performances are obtained using the maximum sidelobe decay windows since in this case the parameters can be estimated by analytical formulas without polynomial approximations [2–6]. The H-term maximum sidelobe decay window ($H \ge 2$) has the most rapidly decaying sidelobes rate, equal to 6(2H − 1) dB/octave, from all the H-term cosine windows [7]. This window is defined as

$$w(m) = \sum_{h=0}^{H-1} (-1)^h a_h \cos\left(2\pi h \frac{m}{M}\right), \quad m = 0, 1, \ldots, M-1, \tag{4}$$

where a_h are the windows coefficients given by [8]

$$a_0 = \frac{C_{2H-2}^{H-1}}{2^{2H-2}}, a_h = \frac{C_{2H-2}^{H-h-1}}{2^{2H-3}}, \quad h = 1, 2, \ldots, H-1, \tag{5}$$

where $C_m^p = \dfrac{m!}{(m-p)!\,p!}$.

Due to the fact that the best performances are obtained using the maximum sidelobe decay windows in the following only these windows are considered.

For $M >> 1$, $W(\cdot)$ can be approximated by [6]

$$W(\lambda) = \frac{M \sin(\pi\lambda)}{2^{2H-2}\pi} e^{-j\frac{(M-1)\pi}{M}\lambda} \frac{(2H-2)!}{\lambda \prod_{h=1}^{H-1}\left(h^2 - \lambda^2\right)}. \tag{6}$$

For estimating the sine-wave parameters by the IpDFT method the ratio α must be firstly evaluated

$$\alpha = \begin{cases} \dfrac{|X_w(l)|}{|X_w(l-1)|}, \text{ if } \; -0.5 \le \delta < 0 \\[2ex] \dfrac{|X_w(l+1)|}{|X_w(l)|}, \text{ if } \; 0 \le \delta < 0.5. \end{cases} \tag{7}$$

From (3) it can be established that

$$\begin{aligned} |X_w(l-1)| &= \frac{A}{2}\left|\left[W(-1-\delta)e^{j\varphi} - W(2l-1+\delta)e^{-j\varphi}\right]\right| \\ |X_w(l)| &= \frac{A}{2}\left|\left[W(-\delta)e^{j\varphi} - W(2l+\delta)e^{-j\varphi}\right]\right| \\ |X_w(l+1)| &= \frac{A}{2}\left|\left[W(1-\delta)e^{j\varphi} - W(2l+1+\delta)e^{-j\varphi}\right]\right|. \end{aligned} \tag{8}$$

For $0 << l << M/2$, the image parts of $X_w(l-1)$, $X_w(l)$ and $X_w(l+1)$ can be neglected and from (6) and (8) α becomes

$$\alpha \cong \begin{cases} \dfrac{H+\delta}{H-1-\delta}, \text{ if } -0.5 \le \delta < 0 \\ \dfrac{H-1+\delta}{H-\delta}, \text{ if } 0 \le \delta < 0.5. \end{cases} \tag{9}$$

From the above expression it follows that δ can be estimated by

$$\hat{\delta} = \begin{cases} \dfrac{(H-1)\alpha - H}{\alpha+1}, \text{ if } -0.5 \le \delta < 0 \\ \dfrac{H\alpha - H + 1}{\alpha+1}, \text{ if } 0 \le \delta < 0.5. \end{cases} \tag{10}$$

From (2) and (10) the frequency f_{in} can be estimated by

$$\hat{f}_{in} = \begin{cases} \left(l + \dfrac{(H-1)\alpha - H}{\alpha+1}\right)\dfrac{f_s}{M}, \text{ if } -0.5 \le \delta < 0 \\ \left(l + \dfrac{H\alpha - H + 1}{\alpha+1}\right)\dfrac{f_s}{M}, \text{ if } 0 \le \delta \le 0.5. \end{cases} \tag{11}$$

Using (6) and (8) the amplitude A can be estimated by

$$\hat{A} = \frac{2^{2H-1}\pi\hat{\delta}\left|X_w(l)\right|}{M\sin\left(\pi\hat{\delta}\right)(2H-2)!}\prod_{h=1}^{H-1}\left(h^2 - \hat{\delta}^2\right). \tag{12}$$

From (3), in which the image part is neglected, the phase φ can be estimated by

$$\hat{\varphi} = \text{Phase}\left\{X_w(l)\right\} - \pi\hat{\delta} + \pi\frac{\hat{\delta}}{M} - \frac{\pi}{2}\text{sign}(\hat{\delta}) - \text{Phase}\left\{W_0\left(-\hat{\delta}\right)\right\}, \tag{13}$$

where: $W_0(\lambda) = \sum_{h=0}^{H-1}(-1)^h \, 0.5a_h \left[\dfrac{\mathrm{e}^{-\mathrm{j}\frac{\pi}{M}h}}{\sin\frac{\pi}{M}(\lambda-h)} + \dfrac{\mathrm{e}^{\mathrm{j}\frac{\pi}{M}h}}{\sin\frac{\pi}{M}(\lambda+h)}\right],$

and sign(·) is the sign function, $\text{sign}\left(\hat{\delta}\right) = \begin{cases} -1, \text{ if } -0.5 \le \hat{\delta} < 0 \\ 1, \text{ if } 0 < \hat{\delta} < 0.5. \end{cases}$

From (11)–(13) it follows that the accuracy of the sine-wave parameters estimates is related on the accuracy of the estimation of δ.

3. **INFLUENCE OF WINDOWING**

In principal, the frequency-domain methods use the windowing approach for reducing the spectral leakage errors [9 – 11].

For a sine-wave without noise the accuracy of δ estimates depends only on the accuracy of the estimation of α by (9). This estimation is affected only by the systematic errors due to the contributions from the image parts of the sine-wave spectrum. These errors depend on the values of l and φ [12].

From (6) the ratio between the magnitudes of the image parts of $X_w(l)$, $|X_{wi}(l)|$ when the (H + 1)-term and H-term maximum sidelobe decay window are used is equal to

$$\frac{|X_{wi}(l)|_{H+1}}{|X_{wi}(l)|_H} = \frac{H(2H-1)}{2\left[(2l+\delta)^2 - H^2\right]}. \tag{14}$$

To avoid the interferences from the DC component it is necessarily to have $l > 2H + 1$ [13]. In this case is evident that $|X_{wi}(l)|_{H+1} < |X_{wi}(l)|_H$. This behaviour is also obtained for the magnitudes of the image parts of $X_w(l-1)$ and $X_w(l+1)$, respectively. Thus, the systematic errors decrease as the window order H increases.

For a sine-wave corrupted by noise if $0 << l << M/2$, it is possible to have systematic errors greater than the errors due to noise (random errors) [14].

On the other hand, the uncertainties of the sine-wave parameter estimates increase as the window order H increases [5]. Thus, in order to obtain accurate estimates it is necessary to choose the window with the lowest H which ensures that the systematic errors are smaller than the errors due noise. In the following such a criterion for choosing the windows is presented.

4. **CRITERION FOR CHOOSING THE WINDOWS**

Due to the digitizing process the quantization noise is always present in sampled signals. Quantization noise is usually assumed to be a uniformly distributed stationary white noise, statistically independent of the input signal.

Let us consider that the quantization noise is generated by an n-bit ADC, with full-scale range FSR. The systematic errors which affect the frequency estimates obtained by the IpDFT method with the H-term maximum sidelobe decay window are smaller than the quantization errors if the following condition is fulfilled [14]

$$\prod_{h=1}^{2H-1}(4l + 2h - 2H - 1) > \mu \frac{2A}{q} \prod_{h=1}^{H-1}\left(4h^2 - 1\right), \tag{15}$$

in which μ is a real number and q is the ADC quantization step ($q = FSR/2^n$).

In order to obtain neglected systematic errors compared with the quantization noise μ must be higher than or equal to 100 [14]. Thus, for μ = 100, the condition (15) established the limit (maximum) value of A, A_{lim} up to which the systematic errors which affect the frequency estimates are neglected compared with the quantization errors. For μ = 100, A_{lim} is given by

$$A_{\text{lim}} = \frac{q}{200} \frac{\prod_{h=1}^{2H-1}(4l + 2h - 2H - 1)}{\prod_{h=1}^{H-1}(4h^2 - 1)}. \tag{16}$$

Based on the above condition a criterion for choosing the windows used in the IpDFT method is proposed.

Criterion. The window which must be used in the IpDFT method is the window with the lowest order which ensures $\hat{A} < A_{\text{lim}}$.

In other words by the proposed criterion the window with the lowest order which ensures that the systematic errors which affect the δ estimates are neglected compared with the quantization errors must be used. Fig. 1 shows the steps involved in the choice of the window by the proposed criterion.

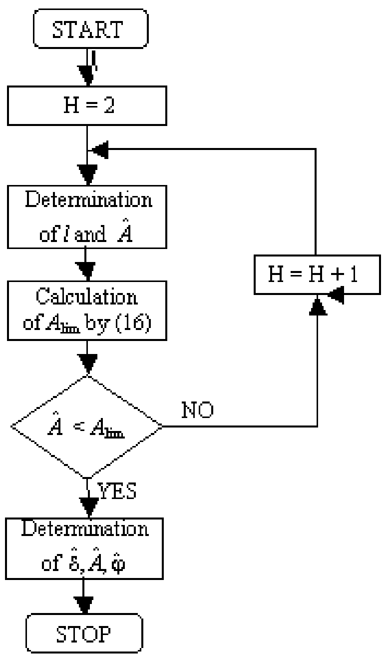
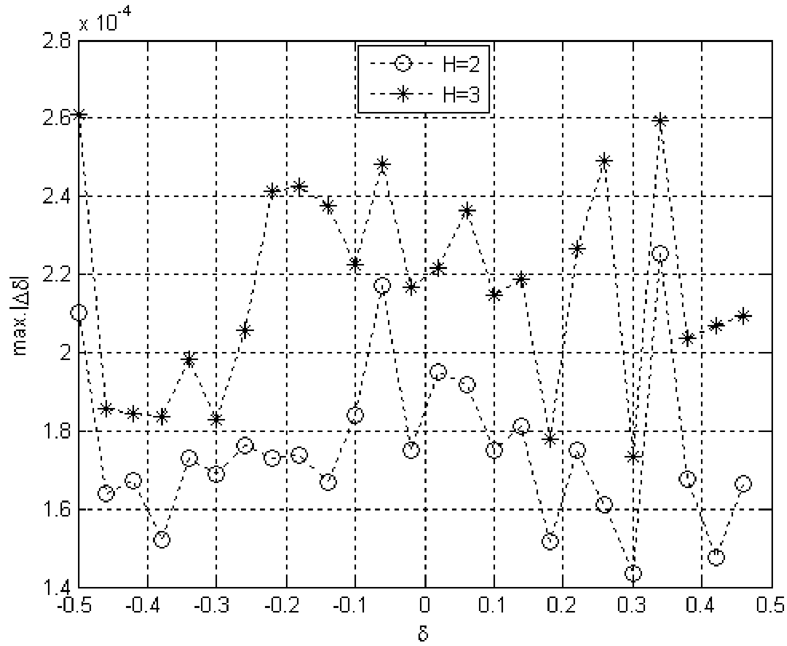


Fig. 1 – The flow chart of the proposed criterion.

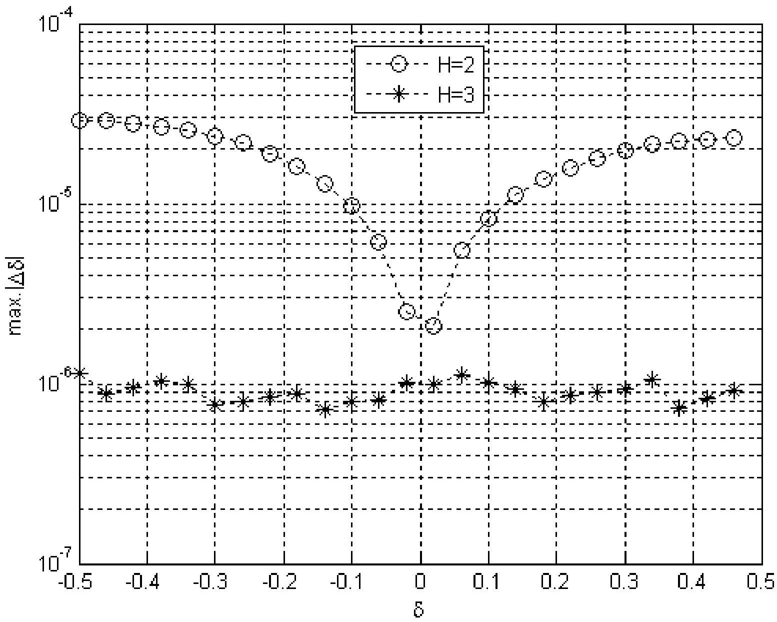
The l value is determined using a search routine applied to the spectral components $|X_w(\lambda)|$ for integer λ values (*i.e.* for $\lambda = 1,\ldots, M/2 - 1$) if the sine-wave frequency signal-to-noise ratio is above threshold [5].

5. COMPUTER SIMULATIONS AND EXPERIMENTAL RESULTS

The aim of this section is to verify the effectiveness of the proposed criterion by means of computer simulations and experimental results. The signal used in simulation is a sine-wave corrupted by the quantization noise of an ideal bipolar n-bit ADC. Quantization noise is modelled by uniformly distributed additive noise. The *FSR* of the ADC is set to 5. Two cases are analyzed. In the first case $l = 12$ and $n = 8$ bits are used and in the second case $l = 14$ and $n = 16$ bits are used. The amplitude A is set to 2 and the phase φ varies in the range $[0, 2\pi)$ rad with a step of $\pi/50$ rad. The record length is $M = 4096$. δ varies in the range $[-0.5, 0.5)$ with a step of 0.04. For each value of δ, $|\Delta\delta|_{max}$ occurring during phase scan is retained. The 2-term and the 3-term maximum sidelobe decay windows are used. Fig. 2 shows for each case $|\Delta\delta|_{max}$ as a function of δ.



a



b

Fig. 2 – $|\Delta\delta|_{max}$ as a function of δ when the 2-term and the 3-term maximum sidelobe decay windows are used for: a) first case; b) second case.

In the first case the proposed criterion established that the 2-term maximum sidelobe decay window performed the best (for $H = 2$ we have $A = 2 < 3.37 = A_{\text{lim}}$) and in the second case established that the 3-term maximum sidelobe decay window performed the best (for $H = 2$ we have $A = 2 > 0.02 = A_{\text{lim}}$ and for $H = 3$ we have $A = 2 < 4.24 = A_{\text{lim}}$). From the results presented in Fig. 2 it follows that the best performances are obtained when the aforementioned windows are used.

The effectiveness of the proposed criterion is verified also by means of experimental results. The test setup is presented in Fig. 3.

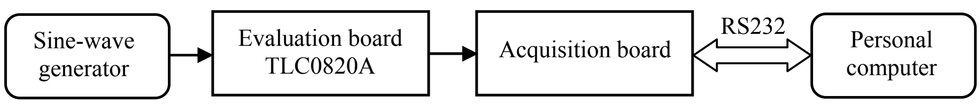


Fig. 3 – The test setup.

The signal generator is Agilent 33220A, which provides very accurate, stable and low distortion sine-waves. TLC0820A is an 8-bit unipolar half-flash ADC. The *FSR* of the ADC is set to 5V. The acquisition board has a 14-bit ADC. Thus, the quantization noise is practically given by the quantization noise of TLC0820A. The sine-waves are characterized by 2 V amplitude and offset and 3.6 kHz frequency. 25 records of $M = 4096$ samples are collected.

The proposed criterion established that the 2-term maximum sidelobe decay window performed the best (for $H = 2$ we have $A = 2 < 3.37 = A_{\text{lim}}$).

Fig. 4 shows the δ estimates for each set when the 2-term and 3-term maximum sidelobe decay windows are used.

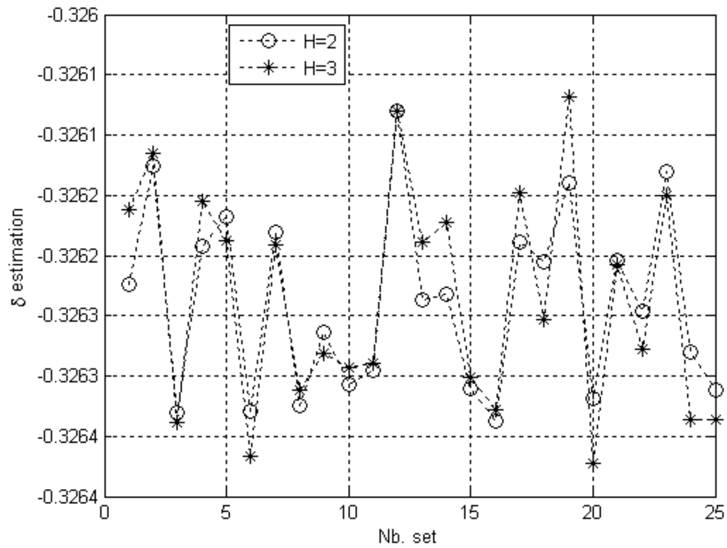


Fig. 4 – δ estimates for each set when the 2-term and the 3-term maximum sidelobe decay windows are used.

For each δ the $\hat{\delta}$ values obtained when these windows are used are close. For each window the same mean of the δ estimates is practically obtained, that is about –0.3263. For $H = 2$ the modulus of the maximum bias is about $1.58 \cdot 10^{-4}$, that is smaller than the one obtained for $H = 3$, which is about $1.69 \cdot 10^{-4}$. Thus, the 2-term maximum sidelobe decay window performed the best. Also, the results obtained are relatively close to the ones obtained by simulation (Fig. 2a).

6. **CONCLUSION**

Unfortunately, in the scientific literature no criterion is given for the optimal choice of the window used in the IpDFT method for estimating the parameters of a sine-wave. Since the best performances of the IpDFT method are obtained when the maximum sidelobe decay windows are used, in this paper a criterion for optimal choice of these windows is proposed. The parameter used for this purpose is the maximum sine-wave amplitude up to which the systematic errors which affect the δ estimates are neglected compared with the quantization errors. The expression of this parameter is given in the paper. The window chosen by the proposed criterion is the window with the lowest order which ensures that the systematic errors which affect the sine-wave parameters estimates are neglected compared with the quantization errors. Computer simulations and experimental results confirm the effectiveness of the proposed criterion.

Received on 17 June, 2008

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