SPECIAL ASPECTS OF MAGNETOHYDRODYNAMIC EQUILIBRIUM CALCULATION FOR DIVERTED TOKAMAK CONFIGURATIONS

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In this paper we present special aspects of calculation the fixed and free boundary MHD equilibria for an axisymmetric divertor tokamak configuration. A "classical" flux coordinate system, amended by a "cast function", has been used. Thus, the unknown moments – the solution of the equilibrium equation – are determined by the difference between the real flux surface contours and those described by the cast function only. With this procedure, the necessary number of moments to describe the flux surfaces in a quite complicated separatrix configuration is small enough to make computations time-efficient. Finally, an exact equilibrium solution to serve as benchmark for the numerical results is presented.

1. INTRODUCTION

In fusion experiments a magnetic field is used to confine plasma in the toroidal vacuum vessel of a Tokamak [1]. The magnetic field is produced by external coils surrounding the vacuum vessel and also by a current circulating in the plasma itself. The resulting magnetic field is helicoidal. Let us denote by \( \mathbf{j} \) the current density in the plasma, by \( \mathbf{B} \) the magnetic field and by \( p \) the kinetic pressure. The momentum equation for the plasma, at the slow resistive diffusion time scale [2] is

\[
\nabla p = \mathbf{j} \times \mathbf{B}.
\]

Taking into account the magnetostatic Maxwell equations (with \( \mu_0 \) the magnetic permeability of the vacuum) we have

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From the equilibrium equation (1.1) it follows that

$$\mathbf{B} \cdot \nabla p = j \cdot \nabla p = 0,$$

(1.3)

\[ i.e., \text{field lines and current lines lie on isobaric surfaces. The innermost magnetic surface (degenerated one) is called magnetic axis, while the plasma boundary corresponds to the surface in contact with a limiter or to a magnetic separatrix (hyperbolic line with an X-point). Rewriting Eqs. (1.1–1.2) under the axisymmetric assumption, we obtain the Grad-Shafranov equation (GSE) [3, 4, 5]. Considering the cylindrical coordinate system } (e_r, e_\phi, e_z) \text{, and the magnetic field } \mathbf{B} \text{ supposed to be independent of the toroidal angle } \phi, \text{ we introduce the poloidal flux}

$$\psi(r, z) = \frac{1}{2\pi} \int_{P_l} \mathbf{B} \cdot \mathbf{ds} = \int_0^r B_r r \, dr,$$

(1.4)
where $' = d/d\psi$. In vacuum, $\Delta^* \psi = 0$.

In the next section we will present some special aspects of equilibrium calculation for diverted tokamak configurations, while in the third section, an exact solution to the GSE will be given.

### 2. CAST FUNCTION PRINCIPLE

Most toroidal flux coordinate systems [6] implicitly assume a nested flux surface structure. However, in a diverted torus the presence of a separatrix breaks this structure and the usual toroidal flux coordinates cannot be used directly. If the plasma boundary is described by a smooth function, the determination of the moments on the boundary (to be used as boundary values by a fixed boundary equilibrium moment solver) is no longer a problem. The presence of an $X$ point implies difficulties; due to the discontinuity of the first derivative, the description of a separatrix by means of Fourier series needs a very high number of moments, implying calculation difficulties. From work performed earlier [7, 8], we know that for a separatrixlike function [as for ASDEX Upgrade, JET (Joint European Torus) or ITER International Thermonuclear Experimental Reactor tokamaks], the expansion in series, based on any set of orthogonal functions, converges slowly near the $X$ point, so it is difficult to separate the high order terms. We have improved our “cast function method” [7] to decrease the number of the essential moments for a diverted toroidal configuration by introducing a function exhibiting the same singularity at the $X$ point. Thus, the separatrix contour is described “exactly” (i.e., with a prescribed accuracy), while the internal real equilibrium flux surface contours are represented approximately. Considering a general curvilinear coordinate system $(a, \omega, \varsigma)$, with $a \in [0, 1]$ an index of the magnetic surfaces, $\omega \in [0, 2\pi]$ a poloidal-angle-like coordinate, and $\varsigma \in [0, 2\pi]$ a toroidal-angle-like coordinate and considering also the cylindrical coordinate system, it is possible to represent the coordinate transformation through Fourier series in $\omega$:

$$
\begin{align*}
    r &= R_{ax} + \rho(a, \omega) \cos \omega, \\
    z &= Z_{ax} + \rho(a, \omega) \sin \omega, \\
    \rho^2(a, \omega) &= a^2 + \text{Re} \left[ \sum_{m=-\infty}^{\infty} \delta_m e^{im\omega} \right],
\end{align*}
$$

(2.1)
where $\rho(a, \omega)$ is the distance between a current point on the plasma boundary and the magnetic axis ($R_{\text{ax}}$, $Z_{\text{ax}}$), while $\delta_m(a)$ are the complex moments ($\bar{\delta}_m = -\delta^*_m$), (the asterisk designating the complex conjugate moments). $a$ has the signification of an equivalent radius.

In the polar representation (2.1) of the separatrix, we have introduced the following cast function $R$ exhibiting the same singularity at the $X$ point [Fig. 1(a)]:

$$
\rho(a, \omega) = R(a, \omega) + A_0(a) + \sum_{k=1}^{\infty} \left[ A_k(a) \cos k\tau + B_k(a) \sin k\tau \right],
$$

$$
R(a, \omega) = a a \left[ 1 + \alpha_1 \nu + \beta_1 \sin \tau - \alpha_0 \sqrt{g(a) + \nu} \right] g(a) = 0.25(a - 1)^2,
$$

where $\nu = 1 - \cos \tau \in [0,2]$, $\tau = \omega - \omega_0$, $\nu = 1 - \cos \tau \in [0,2]$, $\tau = \omega - \omega_0$, with $\omega_0$ the poloidal angle corresponding to the $X$ point. $\nu_0$ is the distance from the magnetic axis to the $X$ point. Considering the separatrix contour given by \(\{r_n, z_n\}\) pairs of points, the geometrical parameters $a_1$, $a_0$, and $b_1$ are then uniquely determined by the geometry of the separatrix, and can be calculated. With 24 $A_k$ and $B_k$ Fourier coefficients, the maximum relative error in $r$ was 0.064% for a particular ASDEX Upgrade configuration. Constant $a$ lines and constant $\omega$ lines are presented in Fig. 1b. Note that these constant lines do not correspond to an equilibrium configuration but to the polar representation (2.2) with the parameters determined by the separatrix geometry only ($a=1$ represents the plasma boundary). The graphical representation from Fig. 1b illustrates how “close” we are with our cast function representation to the real equilibrium constant flux surfaces.

To use this approach in equilibrium calculations, we can write

$$
R^2 = R_0^2 + \bar{R}^2 = R_0^2 + \Re \left[ \sum_m \delta_m e^{im\omega} \right], \quad R_0^2 = \langle R^2 \rangle_{\omega}, \quad \bar{R}^2 = \bar{R}^2 - R_0^2.
$$

With this notation, we can write Eq. (2.1) in the form

$$
\rho^2(a, \omega) = a^2 + \Re \left[ \sum_m \bar{\delta}_m e^{im\omega} \right] + \bar{R}^2, \quad \bar{\delta}_m = \delta_m - \bar{\delta}_m.
$$

Thus, the $\bar{\delta}_m$ moments, to be determined by an equilibrium solver, describe a function of class $C^1$ or higher.
Fig. 1 – a) Illustration of the cast function method for a particular plasma configuration of the ASDEX Upgrade tokamak (shot No. 13476 at 5.2 s). The dashed line represents the real plasma boundary, while the solid one represents the contour given by the cast function \( R(a, \omega) \), only. \( a_1 = 0.996312, a_2 = 0.872609, b_1 = 0.14543, \) and \( a_0 = 1.426115 \). b) constant \( a \) and \( \omega \) lines obtained for the considered diverted configuration by using the representation of Eq. (2.4) only.

3. COMPARISON WITH EXACT EQUILIBRIUM SOLUTIONS

Analytical solutions of the GSE are very useful for theoretical studies of plasma equilibrium, transport, and magnetohydrodynamic stability. These solutions can be used also as a benchmark of numerical codes, but existing exact solutions are very restricted in a variety of allowed current density profiles.

Considering general parabolic dependencies of pressure and poloidal current, the GSE reads

\[
\Delta^* \psi = - \left( a r^2 + a \right) \psi - \left( b r + \frac{B}{r} \right),
\]

representing the most complicated form keeping the GSE linear. The four free parameters allowing to independently specify the plasma current \( I_{\text{pl}} \), the poloidal beta \( \beta_{\text{pol}} \), the internal inductance \( l_i \) and the safety factor at the magnetic axis \( q_{\text{ax}} \) or at the plasma boundary \( q_b \).

We have converted the original inhomogeneous partial differential equation (PDE) into two problems: a homogeneous PDE and an inhomogeneous ordinary differential equation, with the general solution

\[
\psi = \psi_o + \psi_{\text{inh}}, \quad \psi_o = R(r)Z(z).
\]
ψ₀ represents the general homogeneous solution, while ψᵩᵢ is any particular inhomogeneous solution. Thus,

$$R' - \frac{1}{r} R' + (ar^2 + \alpha - k^2) R = 0, \quad Z' + k^2 Z = 0,$$

with $k$ an arbitrary constant. The notation with prime represents the derivative with respect to the independent variable. In the following, for exemplification, we will consider the case $a > 0$ only.

To obtain the general homogeneous solution, we use the new variable $x = \sqrt{ar^2}/2$, obtaining

$$R' + \left(1 - \frac{2\eta}{x}\right) R = 0, \quad \eta = \frac{k^2 - \alpha}{4\sqrt{a}}. \tag{3.4}$$

This equation is the differential equation for the Coulomb wave functions [10, 11] with the Coulomb wave functions $F_L(\eta,x)$ and $G_L(\eta,x)$ as solutions. In our case $L = 0$. Thus, the general solution of the homogeneous equation will be given by

$$\psi_0(r,z) = \left[ C_F \left( \eta, \frac{\sqrt{a}}{2} r^2 \right) + C_G \left( \eta, \frac{\sqrt{a}}{2} r^2 \right) \right] \left[ C_i \cos(kz) + C_4 \sin(kz) \right]. \tag{3.5}$$

To find a particular solution to the inhomogeneous equation, we are seeking for a solution of the form

$$\psi_{\text{inh}}(r,z) = \frac{\beta}{\alpha} + g_{\text{inh}}(r), \quad \text{with } g_{\text{inh}}(x) = x e^{-i\eta} \left( \frac{\beta}{\alpha} - \frac{b}{a} \right), \quad x \equiv \frac{\sqrt{a}}{2} r^2. \tag{3.6}$$

After some tedious calculations, we obtain

$$\psi_{\text{inh}}(r) = \frac{\beta}{\alpha} + i \sqrt{\frac{\alpha}{4}} r^2 e^{-i\eta/3} \sum_{n=0}^{\infty} \frac{1}{(n+1+i\frac{\sqrt{a}}{4\sqrt{a}})^n} \times _2 F_1 \left( n+2,1; n+2+i\frac{\alpha}{4\sqrt{a}} - \frac{1}{2} \right) \left( i \frac{\sqrt{a}}{2} r^2 \right)^n, \tag{3.7}$$
where \( F_1 \) is the hypergeometric function. Thus, the general solution is

\[
\psi(r, z) = [C_1 F_0(\eta, r) + C_2 G_0(\eta, r)] [C_3 \cos \omega + C_4 \sin \omega] + \psi_{\text{app}}(r). \tag{3.8}
\]

Knowing the values of the \( a, b, \alpha, \) and \( \beta \) parameters, the constants \( C_i \) can be determined if both the value of the flux function \( \psi \) on the plasma boundary and the plasma contour are given.

Fig. 2 – Equilibrium parameters calculated analytically for the diverted plasma corresponding to the discharge No. 5000 at 1.55 s of the ASDEX Upgrade tokamak [12], characterized by the parameters: \( I_p = 1 \) MA, \( \beta_{\text{pol}} = 1., \) \( B_{\text{vac}} = 1.687 \) T at \( r_{\text{vac}} = 1.65 \) m, \( l_i = 0.68, \) and \( q_{\text{ax}} = 1.2. \) The correspondent plasma model parameters are: \( a = 2.52, \beta_{\text{pol}} = 0.24, b = 2.0, \beta_{\text{pol}} = 0.21. \) a) constant poloidal flux lines for this configuration are represented. For the following plasma parameters \( I_p = 1 \) MA, \( R_{\text{vac}} = 0.65 \) m, \( l_i = 1., \) and different poloidal beta coefficients, we have obtained: (1) \( \beta_{\text{pol}} = 0.1, \ a = 20.167, b = 0.25380632464345, \) \( \beta = 0.029814802110195; \) (2) \( \beta_{\text{pol}} = 0.4, \ a = 1.1127693929202, b = 0.1265389944282, \) \( \beta = 0.029814802110195; \) (3) \( \beta_{\text{pol}} = 50.8, \ a = 2.8655381387255, b = 0.34574886600619, \) \( \beta = 0.00008394464285; \) (4) \( \beta_{\text{pol}} = 1, \ a = 3.7640700185485, b = 0.3612685457702, \) \( \beta = 0.00008394464285; \) and (5) \( \beta_{\text{pol}} = 2, \ a = 8.2689854561803, b = 0.54189500267279, \) \( \beta = -0.0033968269249; \) b) toroidal current density \( j(r, 0), \) and (c) pressure \( p(r, 0) \) distributions along the \( z = z_{\text{ax}} \) axis.

4. CONCLUSIONS

Because in a diverted torus the presence of a separatrix breaks the structure of the implicitly assumed nested flux surfaces, the usual toroidal coordinates
cannot be used directly. Due to the discontinuity of the first derivative at the \( X \) point, the determination of the moments on the boundary (to be used as boundary values by a fixed boundary equilibrium moment solver) implies difficulties. Our approach consists of introducing a “cast function” which allows us to describe the separatrix exactly (i.e., with a prescribed accuracy), and to represent the internal flux surface contours approximately. The moments to be determined, in order to describe the MHD plasma equilibrium, are related now to the difference between the real flux surface contours and the contours described by the polar representation (2.2).

We have presented an exact solution to the Grad–Shafranov equation having a current density parametrisation with four degrees of freedom. Thus, an independent choice of the plasma current \( I_{pl} \), the poloidal beta \( \beta_{pol} \), the internal inductance \( l_i \), and the safety factor \( q \) at the boundary or at the magnetic axis can be made.

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REFERENCES

2. V. Shafranov, On magnetohydrodynamical equilibrium configurations, Soviet Physics JETP, 6, 3, 1013, 1958.