AN EFFICIENT INTEGRAL METHOD
FOR THE COMPUTATION OF THE BODIES MOTION
IN ELECTROMAGNETIC FIELD

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A method for computing the motion of conducting bodies in electromagnetic field is
presented. The field problem is a 3D eddy current problem and is solved by means of
an integral method. The integral eddy current formulation requires only the
discretization of the current sources domains and conducting bodies and contains no
explicit velocity terms. The mechanical problem is solved by means of a
predictor-corrector method. Both problems are coupled together in order to obtain the
time evolution of the conducting body.

1. INTRODUCTION

This paper presents a method for solving the electro-mechanic coupled
problem in which the motion of bodies takes place under the influence of
electromagnetic forces. An efficient field computation program is very useful for
the fast analysis of a great number of electro-mechanical devices, such as rail
launchers, rotating electrical machines, eddy current breaking systems, magnetic
levitation systems.

The major difficulties in solving such problems arise from the fact that the
solution of the 3D eddy current problem depends on the speed and the position of
the moving bodies while the motion of the bodies depends on the magnetic forces,
thus on the solution of the field problem. As it will be shown next, this solution is
obtained in an iterative manner. At each iteration the position of the moving body
is reevaluated and a new field problem is solved. This can lead to substantial

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execution times. It is important that the methods used for solving such problems obtain as possible an accurate solution in a most efficient manner.

Various methods for solving eddy current problems in which the motion of the bodies is imposed already exist. The finite element method (FEM) popularity stems from the fact that it works with sparse matrices and its implementation is relatively easy. Applying this method for problems with moving bodies, however, poses some difficulties. The main problem is that, with each new position of the analyzed bodies, a new partial or full remeshing is required [1]. This process is often ineffective and can lead to substantial execution times, especially in the case of 3D problems. In order to avoid the costly process of remeshing, various other FEM techniques have been developed. Several of these techniques are outlined in [2].

The hybrid method FEM-BEM has great advantages when dealing with field problems with moving bodies. Because it can work with unbounded domains the air region is not discretized and the bodies can move freely at any distance. The field problem in air domain is solved by using the integral boundary element (BEM) method. The conducting region field problem is solved employing FEM. The resulted two solutions are then coupled together by taking into account the boundary conditions on the surface of the bodies [3].

The methods described in [3–4] do not impose the type of the motion and the conducting bodies move freely under the influence of the magnetic force. In [4] the bodies are treated with FEM in their own local system of coordinates, while the surrounding air is treated with BEM in a global system of reference. This method, however, seems to be laborious due to the fact that the entire system of equations contains explicitly the velocity term. Furthermore, the implementation of the BEM technique for 3D domains might be problematic.

The integral method for 3D eddy current problems first described in [5] has several important advantages. This method only requires the discretization of the conducting bodies and of the sources and also treats unbounded domains. The Maxwell-Hertz equation are written in the local system of coordinates of each body, thus no explicit velocity terms are required. Another important advantage of this method is that it does not require any remeshing for each new position of the bodies. In [6] the method is used for solving a problem with moving bodies, but in this case the motion is imposed. Although the method presented in [5] does not deal with nonlinear media it can be extended by using the polarization fixed point method [7–8]. This iterative method always insures the converge [9–10]. Although its speed of convergence is not as fast as other similar methods, like Newton-Raphson, there are various techniques that can be used to increase it [11].

The method described in this paper couples the integral method’s field solution with the solution of the motion problem. The motion problem is calculated by means of a predictor-corrector method.
2. THE INTEGRAL METHOD

2.1. PROBLEM FORMULATION

Let there be a number of $N$ conducting bodies, each moving with a certain unknown $v_k$ velocity. In the local frame of each conducting body we can write

$$E = -\left( \frac{\partial A}{\partial t} + \text{grad}V \right),$$

where $E$ is the electric field intensity, $A$ is the magnetic vector potential, and $V$ is a scalar potential.

The constitutive relationship of the magnetic field is given by

$$B = \mu_0 H,$$

where $B$ is the flux density, $H$ the magnetic field intensity and $\mu_0$ is the permeability of free space.

From (2) and Ampère’s law we obtain:

$$\nabla \times \nabla \times A = \mu_0 \left( J + J_0 \right),$$

where $J$ is eddy current density and $J_0$ is imposed current density.

The solution for this equation is given by the Biot-Savart formula

$$A = \frac{\mu_0}{4\pi} \int_{\Omega_c} \frac{J}{r} \, dv + A_0,$$

where $A_0$ is the magnetic vector potential given by the imposed currents in the coils.

From (1) and (4) the eddy current integral equation is obtained as

$$\rho J + \frac{\mu_0}{4\pi} \frac{d}{dt} \int_{\Omega_c} \frac{J}{r} \, dv + \text{grad}V = -\frac{dA_0}{dt},$$

where $\rho$ is the electric resistivity. The electric scalar potential $T$ is defined as

$$\nabla \times T = J.$$

The current density normal component is null on the conducting domains $\Omega_c$:

$$J \cdot n = n \cdot \nabla \times T = 0.$$
2.2. NUMERICAL APPROACH

The current density $J$ is expressed in terms of cotree edge shape functions $N_k$ [5]:

$$J = \text{rot} T = \sum_{k=1}^{n} i_k(t) \text{rot} N_k.$$ (8)

By using the Galerkin technique on (5) the weak formulation results in

$$\int_{\Omega} \rho \text{rot} T \cdot \text{rot} N_k \, dv + \frac{\mu_0}{4\pi} \frac{d}{dt} \int_{\Omega} \frac{1}{r} \text{rot} N_k \cdot \text{rot} T \, dv = - \int_{\Omega} \text{rot} N_k \cdot \frac{dA_0}{dt} \, dv.$$ (9)

The final equation can be written as

$$RI + \frac{d}{dt}(LI) = -\frac{d}{dt}\phi,$$ (10)

where the vectors $I$ and $\phi$ are

$$I = (i_1, i_2, \ldots, i_n)^T, \quad \phi = (\phi_1, \phi_2, \ldots, \phi_n)^T.$$ (11)

The field sources have been chosen as thin conducting wires with an imposed current $i$. For these the magnetic vector potential is given by the Biot-Savart law

$$A_0 = \frac{i\mu_0}{4\pi} \int_{\Gamma} \frac{dl}{r}.$$ (12)

Assuming that we have $M$ distinct coils, the $\phi$ entries can be written as

$$\phi_k = \int_{\Omega} \text{rot} N_k \cdot A_0 \, dv = \frac{i\mu_0}{4\pi} \int_{\Omega} \text{rot} N_k \cdot \left(\frac{d}{dt} N_k + \frac{M}{4\pi} \sum_{v=1}^{M} \int_{\Gamma_v} \frac{dl}{r} \right) \, dv.$$ (13)

The entries of the matrices $R$ and $L$ are given by

$$R_{ik} = \int_{\Omega} \rho \text{rot} N_i \cdot \text{rot} N_k \, dv,$$ (14)

$$L_{ik} = \frac{\mu_0}{4\pi} \int_{\Omega} \left(\frac{1}{r} \text{rot} N_i \cdot \text{rot} N_k \right) \, dv.$$ (15)

One of the main advantages of the proposed method is that for each new position of the bodies, only certain parts of $L$ and $\phi$ will change. The $R$ matrix will remain unchanged.
From (15) we can see that the only $L_{ik}$ coefficients that need to be updated are the ones belonging to conducting bodies that have changed their relative position. The $\phi_k$ terms (see (13)) have similar properties.

If the conducting bodies do not change their relative positions the $L$ matrix will remain unchanged. In this case the execution time will be drastically decreased, due to the fact that computing the $L_{ik}$ coefficients is a costly operation.

Equation (10) is solved by integrating it over the time interval $\Delta t = t_1 - t_0$

$$R\int_{t_0}^{t_1} I\, dt + \int_{t_0}^{t_1} d(LF) = -\int_{t_0}^{t_1} d\phi.$$  \hspace{1cm} (16)

Assuming that $I$ has a linear variation between $t_0$ and $t_1$, (16) becomes

$$\frac{\Delta t}{2} R(I_1 + I_0) + L_1I_1 - L_0I_0 = -\phi_1 + \phi_0.$$  \hspace{1cm} (17)

The previous relation holds for any consecutive $t_n$ and $t_{n+1}$ moments and can be rewritten in the following form

$$\left(\frac{\Delta t}{2} R + L_{n+1}\right)I_{n+1} = \left(-\frac{\Delta t}{2} R + L_n\right)I_n - \phi_{n+1} + \phi_n,$$  \hspace{1cm} (18)

where the $I_{n+1}$ vector is the system’s unknown. In order to solve (18) for the current moment $t_{n+1}$ we need to know the solution $I_n$ from the previous moment $t_n$.

3. THE PREDICTOR-CORECTOR METHOD

In this section the predictor-corrector algorithm for integrating Newton’s equations of motion is presented. Only translation is analyzed.

By using this method we can determine the trajectory of a rigid body whose initial velocity and position are known and which moves under the influence of forces which vary in time and space. Only the Oz direction will be analyzed, the results can be easily extended to any other direction.

Let there be a body of mass $m$ for which we know its initial position and speed. Let there be $F$ the varying force that acts upon it. We seek to determine the trajectory of that body after a certain time $T$ has elapsed.

Let there be $\Delta t = t_1 - t_0$ the current time step. We assume that at the $t_0$ moment the body is located at the $z_0$ position, has the velocity $v_0$ and is under the influence of a force $F_0$. Having known all this data we are required to compute the position $z_1$ of the body at the moment $t_1 > t_0$. But if the force that acts upon the body varies with position, in order to compute $z_1$, we need to know beforehand the value of that force at $z_1$, which is not always possible.
The predictor-corrector method solves this predicament by calculating the position at the current time step $\Delta t$ in multiple stages. Firstly, it assumes that the force $F_0$ acting at $t_0$ is constant and based on this predicts a temporary $z_1$. Then reevaluates the force that acts upon the body at $z_1$ and corrects this position into a new $z_1$, by assuming, this time, that the force has a linear variation between $z_0$ and $z_1$. The correction iterations are repeated until the distance between two consecutive corrected positions become sufficiently small. Once the newly position has been determined we can move further to time step $\Delta t = t_2 - t_1$.

In order to determine the expressions needed for applying this method we start off from Newton’s second law of motion and the definition for acceleration and speed

\[
F = ma, \quad (19)
\]
\[
a = \frac{dv}{dt}, \quad (20)
\]
\[
v = \frac{dz}{dt}, \quad (21)
\]

where $F$ is the force that acts upon the body of mass $m$ on the Oz direction.

Let $t_0$ the initial time for which we know $z_0$ and $v_0$ and $t$ be any moment in time with $t > t_0$. Replacing (20) in (19) and integrating we obtain

\[
v(t) = \frac{1}{m} \int_{t_0}^{t} F(\tau) d\tau + v_0. \quad (22)
\]

By solving (21) we can compute the position at the moment $t$ as

\[
z(t) = \int_{t_0}^{t} v(\tau) d\tau + z_0. \quad (23)
\]

We denote with

\[
I = \frac{1}{m} \int_{t_0}^{t} F(\tau) d\tau \text{ and } J = \int_{t_0}^{t} I(\tau) d\tau, \quad (24)
\]

and analytically evaluate the integrals for the two cases.

If the force does not vary in time (24) becomes

\[
I = \frac{1}{m} \int_{t_0}^{t} \tau d\tau = \frac{\Delta t}{2m} F \quad \text{and} \quad J = \int_{t_0}^{t} \frac{1}{m} F(\tau - t_0) d\tau = \frac{\Delta t^2}{2m} F, \quad (25)
\]
where $\Delta t = t - t_0$ is the time step.

If the force has a linear variation in time (24) becomes

$$ I = \frac{1}{m} \int_{t_0}^{t} F(\tau) d\tau = \frac{\Delta t}{2m} (F + F_0) \quad \text{and} \quad J = \frac{1}{6m} [F(t_{n+1}) + 2F(t_n)] \Delta t^2. \quad (26) $$

Relationships (22) and (23) hold for any consecutive $t_n$ and $t_{n+1}$ moments and can be written, considering (24), as:

$$ v_{n+1} = v_n + I \quad \text{and} \quad z_{n+1} = z_n + v_n h + J, \quad (27) $$

where the $n+1$ index denotes the current moment and $n$ the previous moment. In order to compute the speed and the position for the current moment $t_{n+1}$ we need to know their values at the previous moment $t_n$.

### 4. THE COUPLED PROBLEM. THE ALGORITHM

In this section we describe the way the eddy current problem is coupled with the mechanical problem. The algorithm is the following:

1) For the time $t = n \Delta t$ the eddy current problem (18) is solved.
2) The magnetic force $F_0$ for the current position $z_0$ is computed.
3) The position $z_1$ is predicted, using (25), (27), by assuming $F_0$ is constant.
4) The body is translated to $z_1$:
   4.1) The eddy current problem (18) is solved for the current position $z_1$.
   4.2) The magnetic force $F_1$ for the current position $z_1$ is computed.
   4.3) The position $z_1$ is corrected, using (26), (27), by assuming a linear $F_0 - F_1$ variation.
   4.4) The body is translated to $z_1$.
   4.5) The execution continues again form 4.1 until the difference between two consecutive corrected $z_1$ values is small enough.
5) Next step $t = (n+1) \Delta t$.

The number of iterations the predictor-corrector method needs to obtain a solution is usually small, typically two. The magnetic force acting upon the bodies is calculated by integrating Maxwell stress tensor on a close surface around the body.

### 5. NUMERICAL RESULTS – ELECTROMAGNETIC LEVITATION

Electromagnetic levitation occurs when the lifting force, caused by eddy currents inside a conducting body due to varying magnetic field, balances the force of gravity. This is the case for the device presented in Fig. 1.
A cylindrical aluminum ($\sigma = 34 \text{ MS/m}, m = 0.107 \text{ kg}$) plate is located above two thin circular wires through which a certain current passes. All three objects are aligned coaxially. In this case the two thin wires act as an approximation of some coils with certain windings.

Both coils carry a.c. currents in opposite directions. The rms value of the outer coil current is $I_{\text{out}} = 8 \text{ kA}$, while for the inner coil three different values have been chosen $I_{\text{int}_1} = 13 \text{ kA}, I_{\text{int}_2} = 10 \text{ kA}$, and $I_{\text{int}_3} = 8 \text{ kA}$, respectively. The current frequency is $f = 50 \text{ Hz}$. The body’s initial distance to the coils is $\delta = 9.8 \text{ mm}$. The coils are located at position $z = 0$.

At the initial moment ($t_0$) the plate is released and, under the influence of the gravitational force, descends towards the coils. As the plate is getting closer to the field source, eddy currents will be induced inside it due to its motion and the varying imposed currents from the coils. The magnetic force will oppose the gravitational force and will try to elevate the plate. In Fig. 2 we can observe the oscillating behavior of the plate.

As the inner current increases, the displacement of the cylinder will be greater, and the necessary time for its stabilization will increase. The conducting body will have a higher altitude at which it will reach equilibrium as the inner current will increase.

It is worth mentioning that in addition to the oscillating movement represented in Fig. 2, the conducting body also exhibits a vibration motion at a frequency of $f_2 = 100 \text{ Hz}$. 

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**Fig. 1 – Device for magnetic levitation.**
6. CONCLUSIONS

The presented method consists in coupling the eddy current problem with the mechanical problem. The eddy current problem is solved by means of an integral method, while the solution for the mechanical problem is obtained with a predictor-corrector method.

The method does not require the discretization of the air region and allows for an easy treatment of unbounded domains. Although it operates with full matrices, only certain parts of them and the free terms will be updated for each new position of the bodies. The method works especially fast when the conducting bodies don’t change their relative position and move only in relation with the field sources.

Several numerical results have been obtained and interpreted for an illustrative example representing a levitation device.

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