ON QUALITATIVE BEHAVIOURS OF A CLASS OF PIECEWISE-LINEAR CONTROL SYSTEMS. PART II. A CASE STUDY

VIRGINIA ECATERINA OLTEAN

Keywords: Hybrid control systems (HCS), Piecewise-linear (PL) systems, Qualitative models, Simulation.

Qualitative models of complex systems recently emerged as useful tools for behaviour prediction in case of incomplete system knowledge or as a variant to a difficult analytical analysis. A class of piecewise (PL) differential systems can be abstracted to a logical automaton, within a hybrid control systems (HCS) framework from control engineering, and the resulting discrete evolution provides information about the qualitative behaviours of the continuous system. The basic general models and related design problems were presented in Part I. Part II presents, within a case study, a solution to the problem of constructing a logical automaton as a discrete event abstraction of a PL differential system. Relevant qualitative evolutions are successfully checked against simulated trajectories and future research directions are proposed.

1. INTRODUCTION

Starting from a piecewise-linear (PL) differential system, firstly introduced in the biological systems literature [1], this paper proposes a qualitative modelling approach, within a hybrid control systems (HCS) framework from control engineering [2]. The motivation of this model conversion problem and the two basic general models involved in the transformation are presented in the first part of the paper [3].

Summing up, the PL differential system comprises a stable linear part controlled by a nonlinear feedback law, which switches from one constant value to another, when a combination of state variables reach given threshold limits. On the other side, the HCS comprises a discrete event system (DES) that controls a continuous plant through an interface. The threshold sensors in the interface implement a partition of the continuous state space. When the continuous state reaches a partition boundary, a plant-symbol is sent to the DES controller which sends, in response, a control-symbol, instantly converted into a constant control

"Politehnica" University of Bucharest, E-mail: ecaterinaoltean@yahoo.com

value for the plant. The plant coupled to the interface is abstracted to a DES-plant automaton, and, starting from a desired discrete path in the DES-plant, the DES controller is built within the discrete event control theory. Recall that, under given assumptions the construction of the DES-plant automaton, representing Problem 1 in the HCS framework [3], can be performed without need to integrate the state equations of the plant. For the continuous plant, the DES controller coupled to the interface behaves like a switching control law.

More specifically, the goal is to design a qualitative model represented by the DES-plant automaton associated to the PL differential system, with a state space partition inferred from the threshold limits in the expression of the feedback control law. The qualitative behaviours of the PL differential system, corresponding to families of continuous trajectories, parameterised by their initial states arbitrarily placed in partition regions, are studied as discrete evolutions of the DES-plant automaton. The general model conversion problem, designated as Problem 4 in [3], is related to, but distinct from the philosophy of the other three basic problems in the HCS framework and the detail that makes the difference is the presence of the feedback control law in the equations of the given PL differential system.

Part II of the paper presents a solution to Problem 4, within a case study dedicated to a second-order PL differential system. The PL differential model and the basic steps of the qualitative synthesis of the DES-plant automaton are introduced in Chapter 2. In Chapter 3, a formula for the feedback control law, depending only on the inferred state space partition, is proposed for numerical simulation experiments. Relevant qualitative behaviours of the PL differential system are checked against corresponding particular simulated trajectories and, starting from the DES-plant evolutions, the predictability of special continuous behaviours is informally discussed, followed by concluding remarks.

2. THE DES-PLANT MODEL OF A SECOND-ORDER PL SYSTEM

For the construction of the controlled DES-plant model of a PL differential system, satisfying equations (1) or (2) in [3], a state space partition is firstly extracted, from the expressions of the feedback switching law. Then, under assumptions A0-A3, Problem 1 is solved using the construction Criterion [3]. No initial states are specified for the PL differential model, so the quantitative behaviours concern families of continuous trajectories.

For simplicity, but without loss of generality, consider the second-order PL differential model proposed, within a distinct qualitative modelling approach, in [4] and given by
Behaviours of piecewise-linear control systems. 2

\[
\begin{align*}
\dot{x}_1 &= k_1 s^-(x_1, \theta_1^i) s^-(x_2, \theta_2^i) - \alpha_1 x_1 \\
\dot{x}_2 &= k_2 s^-(x_1, \theta_1^i) s^-(x_2, \theta_2^i) - \alpha_2 x_2
\end{align*}
\]

with the parametric constraints

\[
\alpha_i > 0, \ k_i > 0, \ 0 < \theta_1^i < \theta_2^i, \ k_i - \alpha_i \theta_1^i > 0, \ i = 1:2.
\]

Recall that, for \(i, j \in \{1:2\}\), the switching functions

\[
s^-(x_i, \theta_j^i) = \begin{cases} 
1, & x_i < \theta_j^i \\
0, & x_i > \theta_j^i
\end{cases}
\]

depend on the signum of \(x_i - \theta_j^i\), respectively (Table 1), so for the requested state space partition in the HCS, the functionals

\[
h_i(x) = x_1 - \theta_1^i, \ h_2(x) = x_1 - \theta_2^i, \ h_3(x) = x_2 - \theta_1^i, \ h_4(x) = x_2 - \theta_2^i,
\]

appear as a natural choice. In the sequel, it is assumed that \(\theta_1^j = \theta_2^j, \ j = 1:2\).

Rewrite (1) as in equation (2) in [3],

\[
\dot{x} = Ax + u(x), \ A = \text{diag}(-\alpha_1)_{i \in \{1:2\}}, \ u(x) = (u_1(x), u_2(x))^T,
\]

where the components of the nonlinear control law are

\[
\begin{align*}
u_1(x) &= k_1 s^-(x_1, \theta_1^i) s^-(x_2, \theta_2^i), \\
u_2(x) &= k_2 s^-(x_1, \theta_1^i) s^-(x_2, \theta_2^i)
\end{align*}
\]

Substituting (3) in (6) gives the following control values, as shown in Table 1a,

\[
u_1 = (k_1, k_2)^T, \ u_2 = (k_1, 0)^T, \ u_3 = (0, k_2)^T, \ u_4 = (0, 0)^T.
\]

Define \(U = \{u_i \in \mathbb{R}^2 : i = 1:4\}\) and the alphabet \(\tilde{R} = \{r_j : i = 1:4\}\) of control-symbols, with \(\widetilde{U} \sim U\). The system (5)–(6) switches between the linear systems

\[
\dot{x} = f_m(x), \ m = 1:4,
\]

with the vector fields

\[
\begin{align*}
f_1(x) &= (k_1 - \alpha_1 x_1, k_1 - \alpha_2 x_2)^T, \ f_2(x) = (k_1 - \alpha_1 x_1, -\alpha_2 x_2)^T, \\
f_3(x) &= (-\alpha_1 x_1, k_2 - \alpha_2 x_2)^T, \ f_4(x) = (-\alpha_1 x_1, -\alpha_2 x_2)^T,
\end{align*}
\]

corresponding to the control values (7), respectively.
Table 1

a. The values of the switching functions (3) and the components of control values (7).

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$\theta_1^1$</th>
<th>$\theta_1^2$</th>
<th>$x_2$</th>
<th>$\theta_2^1$</th>
<th>$\theta_2^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^+(x_1, \theta_1^1)$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$s^-(x_2, \theta_1^1)$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$u_1(x)$</td>
<td>$k_1$</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^-(x_1, \theta_1^2)$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$s^+(x_2, \theta_1^2)$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$u_2(x)$</td>
<td>$k_2$</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. The signs of the functionals (4) of the state space partition.

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$\theta_1^1$</th>
<th>$\theta_1^2$</th>
<th>$x_2$</th>
<th>$\theta_2^1$</th>
<th>$\theta_2^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_1(x) = x_1 - \theta_1^1$</td>
<td>$-$</td>
<td>0</td>
<td>+</td>
<td>$h_2(x) = x_1 - \theta_1^2$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$h_3(x) = x_2 - \theta_2^1$</td>
<td>$-$</td>
<td>0</td>
<td>+</td>
<td>$h_4(x) = x_2 - \theta_2^2$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

The nonempty intersection of the halfspaces separated by $\ker(h_i)$, $i=1:4$, defines in the state plane nine open rectangles, representing the geometrical images of the partition cells, respectively labelled by the symbols of the DES-plant alphabet of states $\mathcal{P} = \{p_q : q=1:9\}$ (Fig.1a). For $\forall q \in 1:9$, the $q$-th qualitative state is the constant vector $b_q = (b_q^1, b_q^2, b_q^3, b_q^4)'$, given by

$$b_q = (\text{sgn}(h_1(x)), \text{sgn}(h_2(x)), \text{sgn}(h_3(x)), \text{sgn}(h_4(x)))',$$  \hspace{1cm} (10)

with $x$ arbitrary in the cell denoted $p_q \in \mathcal{P}$ (see also (7) and (8) in [3]). Define $B = \{b_1, \ldots, b_q\} \sim \mathcal{P}$ the set of qualitative states. The locations of the vector fields (9) and of the qualitative states (10) are deduced from Table 1, respectively.

Based on the partition and on the qualitative states specified in Fig.1a and applying the Criterion in [3], the DES-plant automaton in Fig.1b is obtained.

Example 1. The transition $p_1 \xrightarrow{r_i} p_2$ takes place, with $r_i \in \mathcal{R}$ associated to $f_i$ (9), because $p_1, p_2 \in \mathcal{P}$ are adjacent on $\ker(h_i)$ and, according to (16) in [3],

$$b_1^i (f_i^i(x) \nabla_x (h_i)) = -(k_1 - \gamma_1 x_1, k_2 - \gamma_2 x_2) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} < 0,$$  \hspace{1cm} (11)

with $b_1^i$ the first component of the qualitative state $b_i$. 
3. QUALITATIVE BEHAVIOURS OF THE PL DIFFERENTIAL SYSTEM
– A COMPARISON TO SIMULATED TRAJECTORIES

An expression of the feedback control law $u(x)$ in (5) is heuristically extracted, as a function of the partition hypersurfaces (4), and implemented in the simulation model. To this purpose, based on the information detailed in Table 1, firstly the switching functions (3) are rewritten as combinations of the signums of the functionals (4).

**Example 2.** Let $s^-(x_1, \theta_1^2) = a \text{sgn}(h_2(x)) + b[1 - \text{sgn}(h_2(x))]$, with $a, b \in \mathbb{R}$ unknown coefficients. From Table 1 results the algebraic system of equations:

\[
\begin{align*}
&\ker(h_1) = (1, 0, 0)^	op, \\
&\ker(h_2) = (1, 0, 0)^	op, \\
&\ker(h_3) = (0, 1, 0)^	op.
\end{align*}
\]

Remark. The assumptions A0-A3 [3] can be tested based on the expressions of the solutions of the differential systems (8)-(9), but this contradicts the attempt to construct the DES-plant automaton without integrating the state equations, i.e. to obtain a qualitative representation based on minimal information. Note that A0 does not hold for the system (1). However, this assumption regards only isolated trajectories, so it is generically true, as illustrated in the simulation experiments.
\[ 1 = -a + 2b, \text{ for } 0 \leq x_1 < \theta_1^2 \text{ and } 0 = a, \text{ for } x_2 > \theta_2^2, \quad (12) \]

with the solution \( a = 0, \ b = 0.5 \), so \( s^- (x_1, \theta_1^2) = 0.5[1 - \text{sgn}(h_2(x))] \).

The procedure is similar for the rest of the switching functions in (3) and the resulting components of the feedback control law in (6) are

\[
\begin{align*}
  u_1(x) &= 0.25k_1[1 - \text{sgn}(h_2(x))][1 - \text{sgn}(h_3(x))], \\
  u_2(x) &= 0.25k_2[1 - \text{sgn}(h_1(x))][1 - \text{sgn}(h_4(x))].
\end{align*}
\]

The simulation model implements equations (5) with the components (13) of \( u(x) \) and the parameters values satisfying the constraints (2): \( k_1 = k_2 = 2.5, \ \theta_1^1 = \theta_1^2 = 1 \) and \( \theta_2^1 = \theta_2^2 = 2 \). The eigenvalues – \( \alpha_1 \) and – \( \alpha_2 \) of \( A \) (5) are kept stable and allowed to vary. The results of simulation experiments, performed in MATLAB, are depicted in Fig. 2, Fig. 3 and Fig. 4, for three distinct values of the ratio \( \alpha_1/\alpha_2 \), respectively.

Despite that fact that the qualitative model in Fig.1b does not capture the trajectories evolving through the intersection of the partition hypersurfaces (as a consequence of \( \text{A0} \) [3]), such evolutions are generated as simulated trajectories. For \( \alpha_1 = \alpha_2 \), this is illustrated by the trajectories started in \( x_{03} \) (Fig. 2a) and \( x_{06} \) (Fig.2b), which both end in \( \ker(h_1) \cap \ker(h_2) \). The presence of this equilibrium point is captured in the qualitative model proposed in [4].

Fig. 2 – Simulated trajectories of the system (5), (13) with \( \alpha_1 = \alpha_2 = 1 \), starting: a) in \( p_1 \), with qualitative evolutions \( p_1(p_2p_3)^* \) for \( x_{01} = (0.4,0)^t \), \( p_1(p_4p_7)^* \) for \( x_{02} = (0,0.4)^t \) and ending in \( \ker(h_1) \cap \ker(h_2) \) for \( x_{03} = (0,1.0,1)^t \); b) in \( p_9 \), with qualitative evolutions \( p_9p_6p_5(p_4p_7)^* \) for \( x_{04} = (2.1,2.5)^t \), \( p_9p_6p_5(p_2p_3)^* \) for \( x_{05} = (2.5,2.1)^t \) and ending in \( \ker(h_1) \cap \ker(h_2) \) for \( x_{06} = (2.5,2.5)^t \).
Behaviours of piecewise-linear control systems. 2

Fig. 3 – Simulated trajectories of the system (5), (13), starting in $p_1$: a) for $\alpha_1 = 1.2 > \alpha_2 = 1$, the qualitative evolutions are: $p_1(p_2p_3)^*$ for $x_{01} = (0.4,0)$ and $x_{03} = (0.1,0.1)^*$, and $p_1(p_4p_7)^*$ for $x_{02} = (0,0.4)^*$; b) for $\alpha_2 = 1.2 > \alpha_1 = 1$, the qualitative evolutions are the same as in a), except $p_1(p_4p_7)^*$ for $x_{03} = (0.1,0.1)^*$.

Depending on the location of the initial states in a given partition cell, the continuous trajectories may transit to distinct adjacent cells. These behaviours appear in the automaton in Fig.1b, as nondeterministic discrete evolutions, like the ones from $p_1$ and $p_9$. Examples of continuous trajectories generating qualitative nondeterminism are those started in $x_{01}$ and $x_{02}$ (Fig.2a) or in $x_{04}$ and $x_{05}$ (Fig.2b). Also, for a given initial state, the continuous evolutions may change their corresponding discrete behaviour with the variation of the ratio $\alpha_1/\alpha_2$, as illustrated by the trajectories started in $x_{03}$ (Fig. 3), or in $x_{06}$ (Fig. 4), respectively.

Fig. 4 – Simulated trajectories of the system (5), (13), starting in $p_1$: a) for $\alpha_1 = 1.2 > \alpha_2 = 1$, the qualitative evolutions are: $p_1(p_2p_3)^*$ for $x_{01} = (0.4,0)^*$ and $x_{03} = (0.1,0.1)^*$, and $p_1(p_4p_7)^*$ for $x_{02} = (0,0.4)^*$; b) for $\alpha_2 = 1.2 > \alpha_1 = 1$, the qualitative evolutions are the same as in a), except $p_1(p_4p_7)^*$ for $x_{03} = (0.1,0.1)^*$. 
The automaton in Fig.1b has two qualitative cycles, \((p_2p_3)^*\) and \((p_4p_7)^*\), reflecting the oscillations along the partition hypersurfaces \(\ker(h_i), \ i = 2,4\), as illustrated in Fig. 3 and Fig. 4.

4. CONCLUSIONS

The DES-plant automaton, constructed, within the HCS framework, from the PL differential system, provides important information for the quantitative behaviours prediction, even in case of incomplete knowledge of the equations parameters. The advantage of this approach comes from the construction Criterion [3], where, technically, deciding whether or not a discrete transition occurs implies only a signum evaluation. Thus, under given assumptions, the challenges of integrating possibly complicated nonlinear differential systems are avoided.

In contrast with the modelling approach in [4], the proposed state space partitioning takes into account only open regions as cells, which drives to a less detailed qualitative model. The price for these simplifications is the risk of possible existence of isolated quantitative behaviours which cannot be predicted from qualitative behaviours.

Algorithmic procedures for DES-plant model construction, in view of behaviour prediction of high order PL models as well as stability analysis of the considered class of PL differential systems using special Lyapunov-based methods [5] are subject of future research. Also, for implementation purposes, it presents interest the study of the class of PL differential systems and of the HCS, with the discontinuous sign function in the feedback law replaced by sigmoid functions.

Received on September 7, 2008

REFERENCES