A DYNAMIC OVERRELAXATION PROCEDURE
FOR SOLVING NONLINEAR PERIODIC FIELD PROBLEMS

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The polarization fixed point method allows the harmonic uncoupling in the treatment of periodic fields in ferromagnetic media, thus exhibiting important advantages with respect to other methods. It always insures the procedure convergence, but the convergence is slow. An efficient technique with an improved rate of convergence, based on the dynamic overrelaxation method, is proposed. The polarization harmonics are directly corrected by the magnetic flux density harmonics. In order to reduce the computational and the memory effort, only the relation $B-M$ between the magnitudes of the field vectors is used.

1. INTRODUCTION

In comparison with the Newton-Raphson method, the polarization fixed-point method (PFPM) [1, 2], employs a magnetic permeability that is maintained constant during the iterations, the nonlinearity of the relationship $B-H$ being transferred to the nonlinear dependence $B-M$. Consequently, the PFPM allows to solve separately for each harmonic of the nonlinear field problems in a periodic regime [3], taking into account the harmonics of the polarization. When a fictitious free space permeability is adopted one can apply simple integral methods for the solution of the electromagnetic field problem at each iteration [4, 5]. This method has allowed the development of an efficient procedure for the analysis of the solidification of ferromagnetic materials in motion [6]. PFPM uses a

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Picard-Banach iterative process whose convergence is insured for a correct choice of the fictitious permeability employed in the computation [7].

Unfortunately, the rate of convergence of the PFPM is smaller and smaller as one approaches the exact solution. In order to increase the rate of convergence an optimal value of the fictitious permeability [2, 7, 8, 9] may be recommended. This technique cannot be utilized when the integral methods are employed to determine the electromagnetic field at each iteration. The convergence can also be improved by applying at each iteration a dynamic overrelaxation [10], [2], which can be implemented independently of the solution method. This procedure is extremely efficient for the analysis of the static fields [11], but for the periodic regime it is inconvenient, since it requires a tremendous computational and memory effort.

A new dynamic overrelaxation technique applied to the computation of periodic fields is proposed in this paper. The polarization harmonics are directly corrected by the magnetic flux density harmonics and the relation $B \cdot M$ between the magnitudes of the field vectors is used.

2. PERIODIC STEADY STATE ELECTROMAGNETIC FIELD PROBLEM

The equations for the quasistationary electromagnetic field in the conductive region $\Omega$ are

$$\nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \times H = \frac{1}{\rho} E + J_0, \quad B = \mu(H + M),$$

(1)

where $E$ is the strength of the induced electric field and $\rho$ is the resistivity. In this case too, for given $\mu$, $\rho$, $J_0$ and boundary conditions, the magnetic flux density is uniquely determined in terms of magnetization, $B = T(M)$, with $T$ being nonexpansive. The field problem is solved separately, for each harmonic [3], [6].

3. POLARIZATION FIXED POINT METHOD

 Instead of nonlinear relationship $H = F(B)$ we use

$$B = \mu(H + M),$$

(2)

where $\mu$ is a constant and $M$ is corrected by the flux density [1, 2]

$$M = \nu B - F(B) = G(B),$$

(3)

$\nu = 1/\mu$ may be chosen such that the function $G$ is a contraction, i.e.
Overrelaxation procedure for solving nonlinear periodic field problems

\[
\|G(B_1) - G(B_2)\|_\mu \leq \lambda \|B_1 - B_2\|_\mu,
\]

where \(\lambda < 1\) and the norm is given by

\[
\|U\|_\nu = \left( \frac{1}{T} \int_0^T \int_\Omega U \cdot (\nu U) \, d\Omega \, dt \right)^{1/2},
\]

with \(T\) being the period and \(\Omega\) the space region. For instance, in the case of isotropic media, the contraction is insured if, at each space point, one chooses \(\mu \in (0.2 \mu_{\text{min}})\) \([7, 2]\), where \(\mu_{\text{min}}\) is the minimum value of the differential permeability in the region considered. Therefore, we may use the vacuum magnetic permeability, when the contraction factor is

\[
\lambda = 1 - \frac{\mu_0}{\mu_{\text{max}}},
\]

where \(\mu_{\text{max}}\) is the maximum value of the differential permeability.

**Picard-Banach scheme.** Starting with an arbitrary initial distribution of magnetization \(M^{(0)}\), the solution of the periodic field problem is obtained through the Picard-Banach scheme

\[
... M^{(n-1)} S T G M^{(n)},
\]

where \(M_n = S(M)\) represents the Fourier expansion of \(M\), truncated to a finite number of terms. The composition \(W = G \circ T \circ S\) is a contraction with a contraction factor \(\lambda\). It has been shown in [3] that the most efficient strategy consists in first retaining only the fundamental harmonic and, then, in refining the solution by introducing successively the higher order harmonics.

**4. DYNAMIC OVERRELAXATION**

After \(n\) iterations, the error with respect to the limit \(M^*_a\) of the sequence in (7) satisfies the relation

\[
\left\| M^{(n)}_a - M^*_a \right\|_\mu \leq \left\| M^{(n+1)}_a - M^{(n)}_a \right\|_\mu / (1 - \lambda).\]

Due to the contraction \(W\) in (8), the error in the field problem solution becomes smaller and smaller with each subsequent iteration. Suppose that \(M^{(n-1)}_a\) is determined following (7), i.e.
\( B^{(a)} = T(M^{(n-1)}_a), \ M^{(a)}_a = \Psi(B^{(a)}) , \)

with \( \Psi \equiv S \circ G \). The overrelaxation is performed by employing a new value \( M^{(a)}_a \) instead of \( M^{(a)}_a \), i.e.

\[
M^{(a)}_a = M^{(a-1)}_a + \theta \Delta M^{(a)}_a ,
\]

where \( \Delta M^{(a)}_a = M^{(a)}_a - M^{(a-1)}_a \) and the overrelaxation factor \( \theta \) is determined to obtain the smallest value of

\[
f(\theta) \equiv \left\| M^{(a+1)}_a - M^{(a)}_a \right\|_\mu^2 = \left\| \Psi \circ T(M^{(a)}_a) - M^{(a)}_a \right\|_\mu^2 .
\]

Since \( T \) is a linear operator, we have

\[
B^{(n+1)} = T(M^{(a)}_a) = B^{(a)} + \theta \Delta B^{(n+1)} ,
\]

with \( \Delta B^{(n+1)} = L(\Delta M^{(a)}_a) \), and (11) becomes

\[
f(\theta) = \left\| \Psi(B^{(a+1)}) - M^{(a)}_a \right\|_\mu^2 .
\]

\( \theta \) is determined by applying the secant method to

\[
f'(\theta) = 2 \left[ \frac{d \Psi}{dB} \right]_{B^{(a+1)}} \Delta B^{(a+1)} - \Delta M^{(a)}_a , \ \Psi(B^{(a+1)}) - M^{(a)}_a \right\|_\mu = 0 ,
\]

where \( \langle \ \rangle \) indicates the time average of the inner product over \( \Omega \) (see (5)) and \( \frac{d \Psi}{dB} \) is the Frechet derivative of the operator \( \Psi \) at \( B^{(a+1)} \).

If the overrelaxation technique is applied only for the fundamental harmonic, the huge volume of numerical calculations and memory necessary to determine \( \Psi \) can be drastically reduced by adopting an approximation for the function \( \Psi \) and its Jacobian. This consists in applying the procedure described in the previous Section to obtain a single table of numerical values of \( \Psi \) calculated for a number of values of \( B \), thus defining a piecewise linear scalar function \( \Psi = h(B) \). The components of the actual vector function \( \Psi \) for the fundamental harmonic are obtained approximately with
5. OVERRELAXATION PROCEDURE FOR SOLVING NONLINEAR PERIODIC FIELD PROBLEMS

\[ f^*(\theta) = 2 \left[ \frac{d}{d B} \mu (B^{(e+1)} - \Delta M^{(e)}_a, \Psi(B^{(n+1)}) - M^{(e)}_a) \right] = 0, \tag{15} \]

\[ \Psi(B) = h(B) B / B, \quad B = (B_1^2 + B_2^2 + B_3^2 + B_4^2)^{1/2}, \tag{16} \]

and the Jacobian with

\[ \frac{d}{d B} \mu = \left( \frac{h'(B)}{B^2} - \frac{h(B)}{B^3} \right) (BB) + \frac{h(B)}{B} I, \tag{17} \]

where \((BB)\) is a dyad and \(I\) is the identity dyadic.

5. ILLUSTRATIVE EXAMPLES

Computation examples are given below to show the efficiency of the proposed overrelaxation techniques as applied to the solution of periodic nonlinear field problems. A field solution at each iteration is derived by using the integral method based on the Green function for an unbounded space (see, e.g., [5]), where the permeability is taken to be everywhere the permeability of free space \(\mu_0\). In the region with nonlinear materials \(\Omega_f\), a discretization grid with \(n_f\) elements \(\omega_k\) is used, within each element the magnetization \(M\) being considered to be constant.

The integral equation employed to obtain the eddy currents at each odd harmonic \(n\) of angular frequency \(\omega_n \equiv n\omega\) is [3]

\[ \rho J_n (r) + \frac{\mu_0}{2\pi} j \omega_n J_n (r') \ln \frac{1}{R} dS' = \]

\[ = -\frac{\mu_0}{2\pi} j \omega_n \left[ \int_{\Omega_0} J_{\omega_n} (r') \ln \frac{1}{R} dS' + \int_{\Omega_f} k \cdot (\nabla' \times M_a (r')) \ln \frac{1}{R} dS' - \right. \]

\[ - \int_{\partial \Omega_f} \ln \frac{1}{R} (M_a (r') \cdot dl') \left] + C_f, \tag{18} \right. \]
where $J_n$ is the current density in the conducting regions of cross section $\Omega$, $J_{0n}$ is the given current density in the nonferromagnetic coil regions of cross section $\Omega_0$, and $C_l$ is a constant for each disjoint conducting region $l$ which is determined by specifying its total current.

For a given harmonic, the matrix associated with (18) remains the same for all the necessary iterations. From each harmonic $n$ of the magnetization, we obtain the $n$-th harmonic of the induced current density by solving (18) and, then, the $n$-th harmonic of the magnetic flux density is calculated from

$$B_n(r) = \frac{\mu_0}{2\pi} k \int_{\Omega} \frac{J_n(r')R}{R^2} dS' + k \int_{\Omega_0} \frac{J_{0n}(r')R}{R^2} dS' +$$

$$+ \int_{\Omega_f} \frac{\nabla \times M_n(r')}{R^2} \times R dS' - k \int_{\Omega_f} \frac{R}{R^2} (M_n(r') \cdot dl') .$$

(19)

![Fig. 1 – Field domain with its ferromagnetic region discretized in 820 elements.](image)
Overrelaxation procedure for solving nonlinear periodic field problems

Consider the cross section of an electromagnetic structure as shown in Fig. 1, where the dimensions are given in mm. The ferromagnetic region is divided in 820 quadrilateral elements. The ferromagnetic material has everywhere the same characteristic $B-H$, as given in Fig. 2, for which the contraction factor is very close...
to unity, $\lambda = 0.99991$. A resistivity $\rho = 10^{-7} \Omega \text{m}$ is taken for the outer cylindrical shell. To test the performance of the modified overrelaxation procedure, the total currents in the four coil sections are (in complex) $I_0$, $-jI_0$, $-I_0$, $jI_0$, as specified in Fig. 1, with the current density constant over each section, such that a rotating magnetic field is produced. Two values for the electric current $I_0$ are chosen, 600 A and 1,500 A, to correspond, respectively, to a weak and to a pronounced saturation. Therefore we consider two cases, namely $ED_1$ and $ED_2$, as indicated in Fig. 3.

To construct the function $\Psi(B)$, necessary in the dynamic overrelaxation technique, we uniformly divided the interval $[0, 2 T]$ in 21 points with positive values of magnetic flux density. For the Fourier analysis employed when determining $\Psi$ and in the iterative PFPM procedure, the functions of time have been approximated by using 320 time steps per period, with a linear variation within each step. The relative error for the fundamental is evaluated as $\|\Delta M^{(n)}\|/\|M^{(n)}\|$ and it is plotted in Fig. 3 versus the computation time for the old PFPM procedure [3] with a constant overrelaxation factor and for the proposed dynamic overrelaxation. In the case of a strong saturation, the old iterative technique cannot be used to decrease the relative error below certain values (for example, below $1.4 \times 10^{-4}$ for a current $I_0 = 1,500$ A, see Fig. 3 b), while the dynamic overrelaxation yields rapidly very small relative errors, e.g., $10^{-12}$ in 240.4 s (case $ED_2$). This shows the excellent performance of this technique for the solution of the periodic regime, even though such a small error is not needed in practice. The harmonic content of the magnetic flux density at the point $P$ in Fig. 1 is shown in Table 1 for the two cases considered in Fig. 3, with the magnetic flux density components given in rms values. To illustrate the overall computation times required for the case of a strong saturation ($I_0 = 1,500$ A), in Table 2, beside the time required for the fundamental harmonic shown in column “1”, obtained by the modified dynamic overrelaxation technique, are given the additional times required when introducing the third harmonic, in column “1+3”, and the fifth harmonic, in column “1+3+5”, the latter two being obtained by the previously proposed procedure [7] with overrelaxation. The numerical results in all the cases considered were generated using a program developed in Visual Fortran 6.6 and employing a 2.5 GHz processor notebook. All the computations were also performed using a refined discretization grid, with 2,100 quadrilateral elements. Only very slight modifications of the results were observed, with the rates of convergence practically in the same relation with respect to each other as in the case of the previous discretization grid.
Table 2

Computation times $ED_2$ in Fig. 3: 1 – dynamic overrelaxation; 2 – PFPM procedure [7] with non-dynamic overrelaxation

<table>
<thead>
<tr>
<th>Case</th>
<th>Harm.</th>
<th>1</th>
<th>1+3</th>
<th>1+3+5</th>
<th>Total</th>
</tr>
</thead>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td></td>
<td>10^{-4}</td>
<td>10^{-3}</td>
<td>10^{-3}</td>
<td></td>
</tr>
<tr>
<td></td>
<td>error</td>
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<td></td>
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<td></td>
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<tr>
<td></td>
<td>t (s)</td>
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<td>17</td>
<td>16</td>
<td>58</td>
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<tr>
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<td></td>
<td>10^{-4}</td>
<td>10^{-3}</td>
<td>10^{-3}</td>
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<tr>
<td></td>
<td>error</td>
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<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>t (s)</td>
<td>262</td>
<td>16.5</td>
<td>16.7</td>
<td>295.2</td>
</tr>
</tbody>
</table>

6. CONCLUSIONS

Application of the PFPM to the analysis of periodic regimes [7, 8] allows the electromagnetic field solution to be obtained for each harmonic separately, which constitutes a particularly important advantage with respect to other methods available in the literature. Replacing the $B$-$M$ characteristic with the relation between the fundamental components of the magnetic flux density and magnetization yields a spectacular reduction of the amount of computation necessary to obtain the fundamental harmonics, by eliminating the necessity to perform the Fourier analysis at each iteration and in each subregion. This, obviously, results in a substantial increase of the rate of convergence of the iterative process. The dynamic overrelaxation procedure further accelerates the convergence. This procedure are efficiently applied to the iterative process associated with the determination of the fundamental harmonics which, due to the smaller contributions of the higher harmonics, brings the field quantities values closer to the actual periodic values. This initial approximation is then corrected by adding higher harmonics and employing the previously developed iterative technique. For strongly nonlinear media, where the contributions of the fundamental and the third harmonics are comparable, the dynamic overrelaxation procedure can be initiated for both these harmonics, but the number of equal segments within the interval considered for the fundamental of the magnetic flux density should be reduced to allow for a number of segments within the interval for the third harmonic [12]. For example, for a similar computational effort as in the cases presented in the previous Section, one can choose about 8 equal segments between $-2T$ and $2T$ for the fundamental, and 4 equal segments between $-0.8T$ and $0.8T$ for the third harmonic.

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