Dedicated to the memory of Prof. Augustin Moraru

FEM-BEM TECHNIQUE FOR SOLVING THE MAGNETIC FIELD IN ELECTRIC MACHINES

PAUL MINCIUNESCU [1], STELIAN MARINESCU [1], IOANA HĂNȚILĂ[2], OANA M. DROSU [3]

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This work presents an efficient iterative FEM-BEM procedure for the computation of the magnetic field in electric machines. The magnetic field is determined separately, by FEM, for a stator and rotor pole pitch, under the assumption that the vector potential satisfies Dirichlet and periodic boundary conditions. The normal derivative of the potential is obtained, which is then introduced into the integral equation written for the air region boundaries, thus yielding the new value of the potential. The numerical expression of the integral equation takes into account the periodicity of the geometrical structure and defines the relationship between the normal derivative and potential only for a polar pitch. The convergence of the procedure is ensured by the large value of the ferromagnetic permeability. For nonlinear media, the large value of the permeability is maintained by applying the polarization fixed point method.

1. **INTRODUCTION**

The hybrid FEM-BEM methods combine the advantages of the finite element method (FEM) with those of the boundary element method (BEM), leading to a extremely efficient method for solving for the magnetic fields in electric machines [1]. Some of the advantages are: the motion of the rotor does not require the reconstruction of the FEM mesh or the use of special techniques for coupling of the rotor and stator meshes, the ferromagnetic media (FEM treated) are separated from the air regions (BEM treated) thus avoiding large permeability discontinuities in the FEM equation, the magnetic field is calculated at any distance, without the need to bound the computational domain.

[1] Research Institute for Electrical Engineering, Spl. Unirii 313, Bucharest, 030138 Romania
[2] The German-Romanian Chamber of Industry and Commerce, E-mail: ihantila@yahoo.com
[3] Department of Electrical Engineering, "Politehnica" University of Bucharest, Spl. Independenței 313, Bucharest, 060042, Romania

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Unfortunately, the rows corresponding to the boundary in the system matrix fill up, the matrix becomes nonsymmetric and nondiagonally dominant, its well-conditioning decreases. Therefore, some computational techniques applicable to sparse matrices can no longer be used.

The iterative FEM-BEM method proposed in this work eliminates this disadvantage. Assuming Dirichlet and periodic boundary conditions, the FEM is used to determine separately the magnetic field of a stator pole pitch and a rotor pole pitch. The matrices of the two equation systems maintain all the advantages of the FEM: they are symmetric, diagonally dominant and their well-conditioning increases due to the Dirichlet boundary condition. The normal derivative of the potential is obtained. Then, by using the BEM on the interfaces between ferromagnetic regions and air, the new value of the potential is obtained from its normal derivative.

The large difference between the value of the permeability of vacuum and that of ferromagnetic bodies ensures the convergence of the iterative procedure. The large value of the permeability is maintained if the nonlinearity of the ferromagnetic media is treated by the polarization fixed point method (PFPM), where the polarization is corrected by the intensity of the magnetic field *H*.

2. THE FEM SOLUTION OF THE MAGNETIC FIELD IN THE STATOR AND THE ROTOR

Assuming a known value for the polarization $\boldsymbol{I}$, the two-dimensional equation of the vector potential written for a stator pole pitch $\boldsymbol{\Omega}_s$ and for a rotor pole pitch $\boldsymbol{\Omega}_r$ (Fig.1a,b) is:

$$-\nabla\cdot\left(\frac{1}{\mu}\nabla A'\right)=\boldsymbol{k}\cdot\left(\nabla\times\left(\frac{1}{\mu}\boldsymbol{I}\right)\right), \tag{1}$$

where $\boldsymbol{k}$ is the unit vector of z-axis. Dirichlet or periodic boundary conditions are imposed. The FEM numerical expressions of equation (1) in the stator and in the rotor are:

$$\boldsymbol{P}_s A_s = -\boldsymbol{P}_s^e A_s^e - \boldsymbol{P}_s^i A_s^i + \boldsymbol{W}_s\,, \tag{2}$$

$$\boldsymbol{P}_r A_r = -\boldsymbol{P}_r^e A_r^e - \boldsymbol{P}_r^i A_r^i + \boldsymbol{W}_r\,, \tag{3}$$

where A_s, A_r are the potentials in regions $\boldsymbol{\Omega}_s$ and $\boldsymbol{\Omega}_r$, respectively, and the interfaces with periodic boundary conditions (PBC), whereas A_s^e, A_s^i, A_r^e and A_r^i are the potentials on the exterior and interior boundaries of these regions: S_s^e, S_s^i,

S_r^e, S_r^i. Elements $p_{j,k}$ of the matrices $\boldsymbol{P}_s$, $\boldsymbol{P}_s^e$, $\boldsymbol{P}_s^i$, $\boldsymbol{P}_r$, $\boldsymbol{P}_r^e$, $\boldsymbol{P}_r^i$ are obtained with the following relationships:

$$p_{j,k} = \int_{\Omega} \mathrm{grad}\varphi_j \cdot \left(\frac{1}{\mu} \mathrm{grad}\varphi_k \right) \mathrm{d}\Omega \,, \tag{4}$$

while the elements of the matrices $\boldsymbol{W}_s$ and $\boldsymbol{W}_r$, are obtained with:

$$w_j = \int_{\Omega} \left(\mathrm{grad}\varphi_j \times \boldsymbol{k} \right) \cdot \left(\frac{1}{\mu} \boldsymbol{I} \right) \mathrm{d}\Omega \,, \tag{5}$$

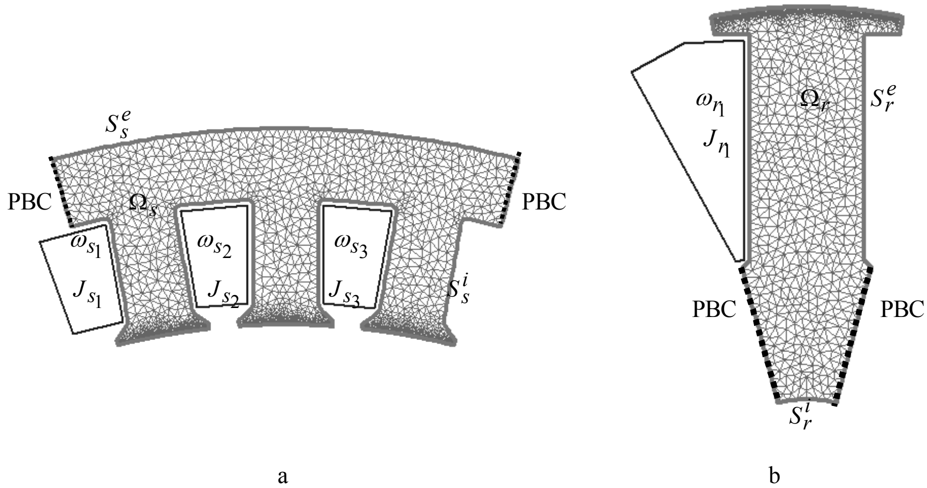

Fig. 1 – Computational regions of one stator pole pitch (a) and one rotor pole pitch (b).

where the weight and trial functions φ_i are first order nodal elements chosen in a mesh with a triangular discretization. Matrices $\boldsymbol{P}_s$ and $\boldsymbol{P}_r$ of the unknowns have the properties of FEM matrices: are sparse, diagonally dominant (assuming Delaunay condition fulfilled) and symmetric. The FEM problems for stator and for the rotor are solved independently. For this work, the Gauss-Seidel method with and over-relaxation factor of 1.9 was preferred.

3. THE BEM CORRECTION OF THE VECTOR POTENTIAL

The integral equation of the vector potential on the boundary $\partial\Omega$ of the air regions Ω_0 is:

$$\lambda A(\boldsymbol{r}) = \oint_{\partial\Omega} \frac{(\boldsymbol{R}\cdot\boldsymbol{n})}{R^2} A(\boldsymbol{r}')\mathrm{d}l' - \oint_{\partial\Omega} \ln\frac{1}{R}\frac{\partial A(\boldsymbol{r}')}{\partial n}\mathrm{d}l' + A_0(\boldsymbol{r}), \tag{6}$$

where $\boldsymbol{r}$ and $\boldsymbol{r}'$ are position vectors of the observation and integration points, $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$, $R = |\boldsymbol{R}|$, $\boldsymbol{n}$ is the exterior normal to $\boldsymbol{r}'$, χ is the angle under which a small neighborhood of Ω_0 is seen from the observation point, defined by $\boldsymbol{r}$ (Fig. 2) and $\boldsymbol{A_0}$ is the vector potential due to the current distributions.

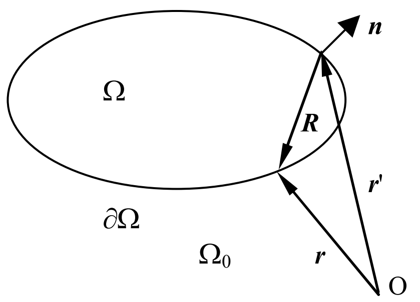

Fig. 2 – The boundary of the air regions.

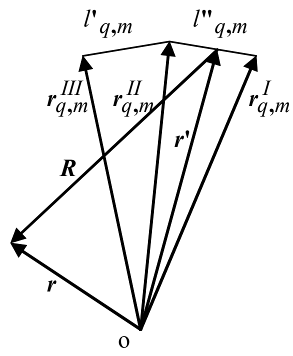

Fig. 3 – The notations used for the calculation of elements $v_{q,k}$.

The boundary is approximated with a polygonal line, where the vector potential is taken to have a linear variation on each boundary element with a constant normal derivative. If we take into account the pole periodicity, equation (6), written for the exterior region to the stator, becomes:

$$V_s^e A_s^e = U_s^e \frac{\partial A_s^e}{\partial n}, \tag{7}$$

where the entries of the matrices V_s^e and U_s^e are computed with (see Fig. 3):

$$v_{k,q} = \chi\delta_{k,q} + \sum_{m=0}^{2p} (-1)^m \alpha(\boldsymbol{r}_k, \boldsymbol{r}_{q,m}^{I}, \boldsymbol{r}_{q,m}^{II}, \boldsymbol{r}_{q,m}^{III}), \tag{8}$$

$$u_{k,q} = \sum_{m=0}^{2p} (-1)^m \beta(\boldsymbol{r}_k, \boldsymbol{r}_{q,m}^{I}, \boldsymbol{r}_{q,m}^{II}), \tag{9}$$

$2p$ being the number of pole pitches, $\delta_{k,q} = \begin{cases} 0, \text{ for } k \neq q \\ 1, \text{ for } k = q \end{cases}$, and

$$\boldsymbol{r}_{q,m}=\begin{pmatrix}x_{q,m}\\ y_{q,m}\end{pmatrix}=\begin{pmatrix}\cos(2m\pi)/(2p) & -\sin(2m\pi)/(2p)\\ \sin(2m\pi)/(2p) & \cos(2m\pi)/(2p)\end{pmatrix}\begin{pmatrix}x_{q,0}\\ y_{q,0}\end{pmatrix}. \tag{10}$$

Functions α and β are defined by:

$$\alpha(\boldsymbol{r},\boldsymbol{r}^{I},\boldsymbol{r}^{II},\boldsymbol{r}^{III})=\int_{l'}\frac{(\boldsymbol{R}\cdot\boldsymbol{n})}{R^2}\times\frac{\left|\boldsymbol{r}'-\boldsymbol{r}^{I}\right|}{\left|\boldsymbol{r}^{II}-\boldsymbol{r}^{I}\right|}\mathrm{d}l+\int_{l''}\frac{(\boldsymbol{R}\cdot\boldsymbol{n})}{R^2}\times\frac{\left|\boldsymbol{r}'-\boldsymbol{r}^{III}\right|}{\left|\boldsymbol{r}^{II}-\boldsymbol{r}^{III}\right|}\mathrm{d}l\,, \tag{11}$$

$$\beta(\boldsymbol{r},\boldsymbol{r}^{I},\boldsymbol{r}^{II})=\int_{l'}\ln\left(\frac{1}{R}\right)\mathrm{d}l\,. \tag{12}$$

Analytical expressions can be found for these functions. From equation (7) we obtain:

$$A_s^e=Z_s^e\frac{\partial A_s^e}{\partial n}\,, \tag{13}$$

with $\left(V_s^e\right)^{-1}U_s^e$. Similarly, for the interior region of the rotor:

$$A_r^i=Z_r^i\frac{\partial A_r^i}{\partial n}\,, \tag{14}$$

with $\left(V_r^i\right)^{-1}U_r^i$**.** For the region between stator and rotor, equation (6) becomes:

$$\begin{pmatrix}V_{ss} & V_{sr}\\ V_{rs} & V_{rr}\end{pmatrix}\begin{pmatrix}A_s^i\\ A_r^e\end{pmatrix}=\begin{pmatrix}U_{ss} & U_{sr}\\ U_{rs} & U_{rr}\end{pmatrix}\begin{pmatrix}\partial A_s^i/\partial n\\ \partial A_r^e/\partial n\end{pmatrix}+\begin{pmatrix}C_{ss} & C_{sr}\\ C_{rs} & C_{rr}\end{pmatrix}\begin{pmatrix}J_s\\ J_r\end{pmatrix}, \tag{15}$$

where $\boldsymbol{J_s}$ and $\boldsymbol{J_r}$ are the matrices of the current density in the stator coil and the rotor coil of one pole pitch, the elements of the matrices V_{ss}, V_{rr}, U_{ss}, U_{rr} are determined with relationships (8), (9), whereas for the computation of matrices V_{sr}, U_{sr} and V_{rs}, U_{rs} for angle $(2m\pi)/(2p)$ in expressions (10) we add and subtract, respectively, angle ξ between the stator and rotor reference. The elements of the matrix C_{ss} are determined with:

$$c_{k,q}=\sum_{m=0}^{2p}(-1)^m\gamma(\boldsymbol{r}_k,\boldsymbol{r}_{q,m}^{I},\boldsymbol{r}_{q,m}^{II},\ldots)\,, \tag{16}$$

where $\boldsymbol{r}_{q,0}^{I},\boldsymbol{r}_{q,0}^{II},\ldots$ are the coordinates of the points on the boundary of the polygonal area ω_{s_q} (Fig.1), where the stator current density J_{s_q} exists. The coordinates $\boldsymbol{r}^{I}_{q,m}, \boldsymbol{r}^{II}_{q,m}, \ldots$ for the homologous points on the other pole pitch are determined with relationships (10). Function γ is defined by:

$$\gamma(\boldsymbol{r}, \boldsymbol{r}^{I}, \boldsymbol{r}^{II}, \ldots) = \mu_0 \int_{\omega} \ln \frac{1}{R} \mathrm{d}\omega \, . \tag{17}$$

The elements of the matrix C_{rr} are determined by relationships similar to (16), whereas in case of the elements of matrices C_{sr} and C_{rs}, to angle $(2m\pi)/(2p)$ in expressions (10) we add and respectively, subtract, angle ξ between the stator and rotor reference. From relationship (15) we obtain:

$$A_s^i = Z_{ss} \frac{\partial A_s^i}{\partial n} + Z_{sr} \frac{\partial A_r^e}{\partial n} + T_{ss} \boldsymbol{J}_s + T_{sr} \boldsymbol{J}_r \, , \tag{18}$$

$$A_r^e = Z_{rr} \frac{\partial A_r^e}{\partial n} + Z_{rs} \frac{\partial A_s^i}{\partial n} + T_{rr} \boldsymbol{J}_r + T_{rs} \boldsymbol{J}_s \, , \tag{19}$$

where

$$\begin{pmatrix} Z_{ss} & Z_{sr} \\ Z_{rs} & Z_{rr} \end{pmatrix} = \begin{pmatrix} V_{ss} & V_{sr} \\ V_{rs} & V_{rr} \end{pmatrix}^{-1} \begin{pmatrix} U_{ss} & U_{sr} \\ U_{rs} & U_{rr} \end{pmatrix}, \tag{20}$$

$$\begin{pmatrix} T_{ss} & T_{sr} \\ T_{rs} & T_{rr} \end{pmatrix} = \begin{pmatrix} V_{ss} & V_{sr} \\ V_{rs} & V_{rr} \end{pmatrix}^{-1} \begin{pmatrix} C_{ss} & C_{sr} \\ C_{rs} & C_{rr} \end{pmatrix}. \tag{21}$$

To account for the motion of the rotor, the rotor/stator coupling matrices are recalculated, where the values of the elements are influenced by angle ξ: V_{sr}, U_{sr}, V_{rs}, U_{rs}, C_{sr} and C_{rs}, as well as matrices $\boldsymbol{Z}$ and $\boldsymbol{T}$.

4. THE ITERATIVE FEM-BEM PROCEDURE

The tangential component of the magnetic field intensity of $\boldsymbol{H}$ is conserved on the boundary $\partial\Omega_0$:

$$\frac{\partial A}{\partial n} = \frac{1}{\mu_r} \left(\frac{\partial A}{\partial n} - I_t \right), \tag{22}$$

where I_t is the tangential component of the polarization.

The iterative FEM-BEM procedure is:

a) Using $\partial A_s^e/\partial n$, $\partial A_r^i/\partial n$, $\partial A_s^i/\partial n$, $\partial A_r^e/\partial n$, we obtain the potentials A_s^e, A_r^i, A_s^i, A_r^e, on the surfaces S_s^e, S_s^i, S_r^e, S_r^i with relationships (13), (14), (18), (19) (BEM);

b) Using the Dirichlet boundary conditions A_s^e and A_s^i on the stator boundary, we solve with FEM the magnetic field in the stator (2) and we obtain $\partial A_s^e/\partial n$ and $\partial A_s^i/\partial n$ on the boundary of the stator pole pitch. Similarly, using A_r^e and A_r^i known on the rotor boundary, by solving equation (3) with FEM we get $\partial A_r^e/\partial n$ and $\partial A_r^i/\partial n$ on the boundary of the rotor pole pitch.

c) We determine the values of $\partial A/\partial n$ on the interfaces of the air regions, by using expression (22).

5. TREATMENT OF NON-LINEARITY

The nonlinear relationship $\boldsymbol{B} = F(\boldsymbol{H})$ is replaced by:

$$\boldsymbol{B} = \mu \boldsymbol{H} + \boldsymbol{I}, \tag{23}$$

where the magnetic permeability μ is constant and the term $\boldsymbol{I}$, which has the significance of polarization, is corrected iteratively by [2, 3]:

$$\boldsymbol{I} = F(\boldsymbol{H}) - \mu \boldsymbol{H} \equiv \mathrm{G}(\boldsymbol{H}). \tag{24}$$

μ can be chosen such that function $\boldsymbol{G}$ is a contraction:

$$\left\| \mathbf{G}(\boldsymbol{B}_1) - \mathbf{G}(\boldsymbol{B}_2) \right\| \le \lambda \left\| \boldsymbol{B}_1 - \boldsymbol{B}_2 \right\|, \tag{25}$$

where $\lambda < 1$ and the norm is defined by:

$$\left\| \boldsymbol{B} \right\| = \sqrt{\int_{\Omega_{fe}} \nu \boldsymbol{B}^2 \mathrm{d}\Omega}\,. \tag{26}$$

Ω_{fe} represents the ferromagnetic region. For example, in the case of an isotropic medium, where $B = f(H)$, we have $I = f(H) - \mu H \equiv g(H)$, and $\boldsymbol{I} = g(H)\boldsymbol{B}/B$ and [4]:

$$\mu > \frac{1}{2}\left(\mathrm{d}f/\mathrm{d}H\right)_{\max}, \tag{27}$$

where $\left(\mathrm{d}f/\mathrm{d}H\right)_{\max}$ is the maximum value of the derivative of the magnetic permeability.

6. **EXAMPLE**

The method presented in this paper was applied in order to determine the magnetic field and no load *emf* for a synchronous wind generator with 12 poles and 36 slots. The *B*-*H* characteristic of the ferromagnetic media is shown in Fig. 4.

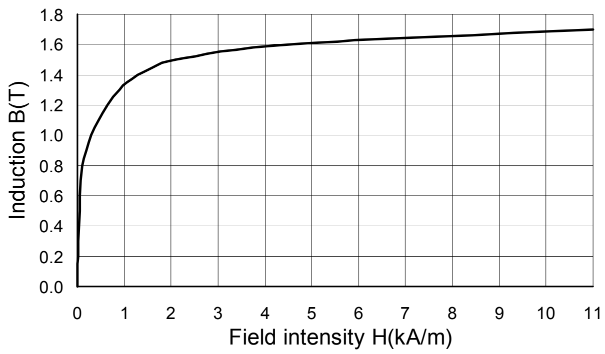

Fig. 4 – *B*-*H* characteristic.

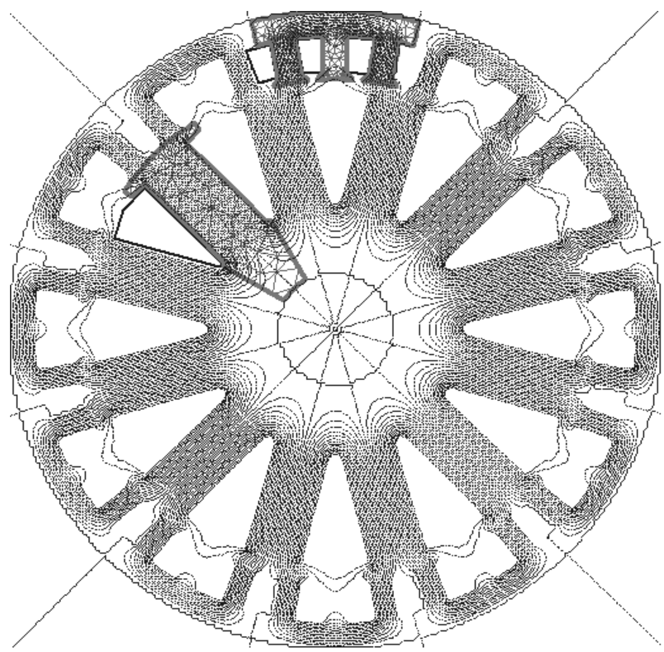

Fig. 5 – Periodiciy of the field lines.

For the ferromagnetic region of one stator pole pitch the FEM mesh consists of 4,015 triangles and 2,234 nodes, where 451 nodes belong to the interface with the air regions. The FEM mesh of a rotor pole pitch consists of 2,448 triangles and 1,356 nodes, of which 262 are in the neighbourhood of air regions.

Fig. 5 show the field lines of the magnetic field induction. The induction in the air gap is presented for three values of the excitation current; the variation in time of the zero load *emf* is also calculated and measured (Fig. 6 and 7).

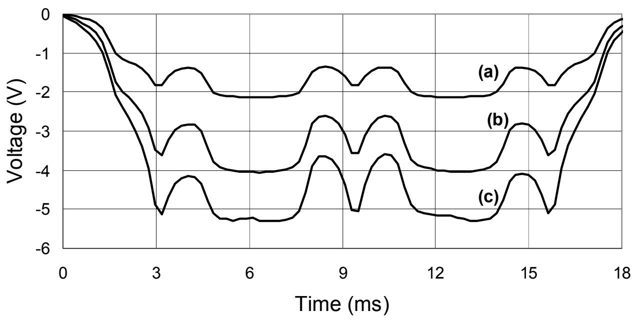

Fig. 6 – Calculated *emf* for rotor currents of: a) 2A; b) 4A; c) 6A.

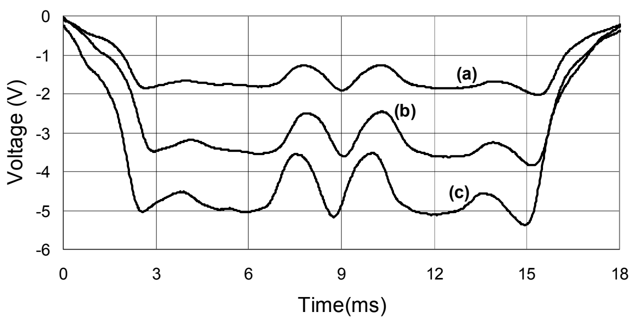

Fig. 7 – Measured *emf* for rotor currents of: a) 2A; b) 4A; c) 6A.

7. **CONCLUSIONS**

The error bound set for stopping the FEM-BEM iterative procedure is:

$$\text{er} < \text{Max}(ER_{BEM}(A), ER_{BEM}(\partial A/\partial n)) \tag{28}$$

where

$$\mathrm{ER}_{\mathrm{BEM}}(A) = \sqrt{\oint_{\partial\Omega_0} (A_n - A_{n-1})^2 \mathrm{d}l} \Bigg/ \sqrt{\oint_{\partial\Omega_0} A_n^2 \mathrm{d}l} \,. \tag{29}$$

The process converges very rapidly: 1–3 iterations for er < $6\mathrm{e}^{-8}$.

The solution of the FEM problem was obtained by the Gauss-Seidel method, where approximately 60 and 50 iterations were needed for the stator region and for the rotor region, respectively, the error bound being:

$$\mathrm{ER}_{\mathrm{FEM}}(A) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(A_{ni} - A_{(n-1)i})^2} \Bigg/ \sqrt{\frac{1}{N}\sum_{i+1}^{N} A_{n_i}^2} < 10^{-7} \,. \tag{30}$$

We need about 20 PFPM iterations for an error limit:

$$\mathrm{ER}_{\mathrm{I-B}}(A) = \sqrt{\int_{\Omega}(I_n - I_{n-1})^2 \mathrm{d}S} \Bigg/ \sqrt{\int_{\Omega} I_n{}^2 \mathrm{d}S} < 10^{-7} \,. \tag{31}$$

Computation time was 2.87s on a notebook having a 2.5 GHz processor.

One of the advantages of the FEM-BEM process consists of the fact that the equations of the magnetic field are satisfied in the air regions. It is enough to calculate the magnetic field separately for a rotor pole pitch and a stator one in order to obtaine the magnetic field in whole machine (Fig. 5).

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REFERENCES

1. M.B. Fauchez, *Magnetic analysis of a switched reluctance motor using a boundary element-finite element coupling method*, IEEE Transactions on Magnetics, **24**, *3*, pp. 627–630, 475–478 (1996).
2. I.F. Hănţilă, *A method for solving stationary magnetic field in nonlinear media*, Rev. Roum. Sci. Techn. – Electrotechn. et Energ., **20**, *3*, pp. 397–407 (1975).
3. F.I. Hănţilă, G. Preda, and M. Vasiliu, *Polarization method for static fields*, IEEE Trans. Magn., **36**, *4*, pp. 672–675 (2000).
4. F.I. Hănţilă, *Mathematical models of the relation between B and H for nonlinear media*, Rev. Roum. Sci. Techn. – Électrotechn. et Énerg., **19**, *3*, pp. 429–448 (1974).