

Dedicated to the memory of Prof. Augustin Moraru

A NEW VECTOR BOUNDARY ELEMENTS PROCEDURE FOR INDUCTANCE COMPUTATION

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This paper presents a new method for solving the integral equation of magnetic vector potential on the surfaces of perfect conductor bodies. The definition of magnetic flux of ideal coils implies that these are perfect conductors, so the normal component of magnetic flux density on their surfaces is equal to zero, which is equivalent with enforcing a zero normal component of magnetic vector potential A and the circulations of A on any closed path on their boundaries, only those surrounding the holes being equal to magnetic fluxes. A and curl A are described by a linear combination of special vector functions associated with the surface nodes. The integral equation is projected on these functions, as well as on another set of specialized functions orthogonal on the first set. The small number of unknowns is a great advantage of the proposed method.

1. INTRODUCTION

Compared with the finite elements method (FEM), the boundary elements method (BEM) based procedures have some important benefits: are applied to unbounded domains, avoid the spurious forces in air regions, have a small number of unknowns associated with a variety of smaller size that compensates for the fact that the system matrix is not sparse. The main drawback is the lack of direct application in heterogeneous media. But, in the structures with piecewise homogeneous subdomains, the interface integral equations can be added [1] and in the case of nonlinear media the hybrid FEM-BEM methods can be used [2, 3].

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The BEM procedures in scalar potential has the advantage of using a relatively small number of unknowns, but great difficulties emerge in the case of multiply connected domains, where integrals singularities require special treatments [4].

This form of the integral equation with the magnetic vector potential as unknown on boundaries is most commonly used [5]

$$\alpha \mathbf{A}(\mathbf{r}) = - \int_{\partial\Omega} \left[\frac{\mathbf{n}'}{R} \times (\nabla' \times \mathbf{A}(\mathbf{r}')) - \frac{\mathbf{R}}{R^3} \times (\mathbf{n}' \times \mathbf{A}(\mathbf{r}')) + \frac{R}{R^3} (\mathbf{n}' \cdot \mathbf{A}(\mathbf{r}')) \right] dS' + 4\pi \mathbf{A}_0(\mathbf{r}), \quad (1)$$

where $\partial\Omega$ is the boundary of homogeneous domain Ω , α is the solid angle under which a small neighbourhood of Ω is seen from the observation point, \mathbf{r} and \mathbf{r}' are the position vectors of observation and source points, respectively, $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, $R = |\mathbf{R}|$, \mathbf{n}' is the normal unit vector, and \mathbf{A}_0 is the vector potential produced by the given distribution of sources in Ω .

The tangential component A_t and the gauge condition $\nabla \cdot \mathbf{A} = 0$ uniquely determine the normal component A_n and the tangential component of the magnetic flux density $(\nabla \times \mathbf{A})_t$, equation (1) being based upon these unknowns. In addition, the treatment of the singularity in the integral of the third term from the right hand of (1) presents some difficulties [6] (only the singularity of the second term is convergent). From the physical point of view, the enforcing of the tangential component A_t of the vector potential is too severe in order to define the normal component of magnetic flux density. In [7] the restriction $A_n = 0$ is proposed, which is consistent with the gauge condition, and the A_t condition is "weakened" by imposing natural circulations of \mathbf{A} (actually A_t) on any closed path on the boundary surface $\partial\Omega$, which is equivalent to imposing the normal component of magnetic flux density. It can be shown that these conditions determine uniquely the tangential components A_t and $(\nabla \times \mathbf{A})_t$. Integral equation (1) has now the following form:

$$\alpha \mathbf{A}(\mathbf{r}) = - \int_{\partial\Omega} \left[\frac{\mathbf{n}'}{R} \times (\nabla' \times \mathbf{A}(\mathbf{r}')) - \frac{\mathbf{R}}{R^3} \times (\mathbf{n}' \times \mathbf{A}(\mathbf{r}')) \right] dS' + 4\pi \mathbf{A}_0(\mathbf{r}). \quad (2)$$

It can be observed that in the equation (2) the right hand third term of (1) does not appear.

By approximating $\partial\Omega$ with a polyhedral surface with triangular facets,

special edge elements have been proposed in [7] in order to describe \mathbf{A}_t . The number of unknowns corresponds only to the number of edge elements associated with a tree resulted from a tree-co-tree spanning. The edge values of \mathbf{A}_t on all surface mesh sides result from the circulations of \mathbf{A}_t on all loops defined by the co-tree. The tangential component $(\nabla \times \mathbf{A})_t$ is defined in local coordinates of each triangular facet.

By projecting the equation (2) on a set of linear independent specialized shape functions obtained from the gradients of the nodal element functions, a different numerical solution of equation (2) is proposed in this paper. This procedure has significant advantages over the method proposed in [7], especially in the case of multiply connected regions (i.e. perfect conductor coils).

2. SIMPLY CONNECTED REGIONS TREATMENT

In the case of perfect conductor domains whose boundaries are not crossed by electric currents, the circulations of \mathbf{A} and $\nabla \times \mathbf{A}$ on any closed path on $\partial\Omega$ are equal to zero. We use the following representation of these vectors

$$\nabla \times \mathbf{A} = \sum_{i=1}^{N'} \alpha_i \mathbf{V}_i, \quad \mathbf{A} = \sum_{i=1}^{N'} \beta_i \mathbf{V}_i, \quad (3)$$

with $N' = N - 1$, N being the number of nodes of the discretization mesh over the surface $\partial\Omega$, α_i , β_i , $i = 1, 2, \dots, N'$, being the unknowns and the vector function \mathbf{V}_i , associated to the node i , having on the triangular facet (p) containing the node i the following form (see Fig. 1)

$$\mathbf{V}_i^{(p)} = \frac{\mathbf{n}_p \times \mathbf{l}_i^{(p)}}{2S_p}, \quad (4)$$

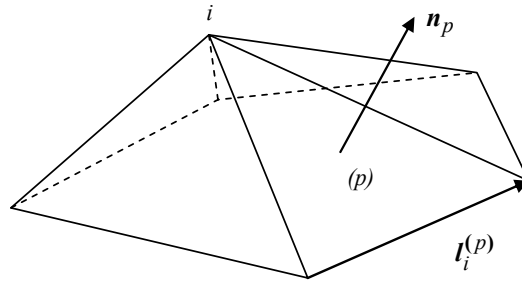


Fig. 1 – The facet (p) associated to the node i .

\mathbf{n}_p and S_p being the normal unit vector and the area of triangular facet (p), and $\mathbf{l}_i^{(p)}$ the length vector of the opposite side of i oriented accordingly with the orientation of \mathbf{n}_p .

Firstly, by projecting (2) on the vector functions V_i , we obtain N' equations

$$\sum_{i=1}^{N'} a'_{ki} \alpha_i + \sum_{i=1}^{N'} b'_{ki} \beta_i = c'_k, \quad k = 1, 2, \dots, N', \quad (5)$$

where

$$a'_{ki} = \int_{\partial\Omega} \int_{\partial\Omega} \frac{1}{R} V_k(\mathbf{r}) \cdot (\mathbf{n}' \times V_i(\mathbf{r}')) dS dS', \quad (6)$$

$$b'_{ki} = 2\pi \int_{\partial\Omega} V_k(\mathbf{r}) \cdot V_i(\mathbf{r}) dS, \quad (7)$$

$$- \int_{\partial\Omega} \int_{\partial\Omega} \frac{1}{R^3} V_k(\mathbf{r}) \cdot [\mathbf{R} \times (\mathbf{n}' \times V_i(\mathbf{r}'))] dS dS',$$

$$c'_k = 4\pi \int_{\partial\Omega} A_0(\mathbf{r}) \cdot V_k(\mathbf{r}) dS. \quad (8)$$

Secondly, the remaining N' equations are obtained by projecting (2) on another set of vector functions U_i , defined by [8], associated also with each node i

$$U_i^{(p)} = \mathbf{l}_i^{(p)} / (2S_p), \quad (9)$$

and we obtain

$$\sum_{i=1}^{N'} a''_{ki} \alpha_i + \sum_{i=1}^{N'} b''_{ki} \beta_i = c''_k, \quad k = 1, 2, \dots, N', \quad (10)$$

where

$$a''_{ki} = \int_{\partial\Omega} \int_{\partial\Omega} \frac{1}{R} U_k(\mathbf{r}) \cdot (\mathbf{n}' \times V_i(\mathbf{r}')) dS dS', \quad (11)$$

$$b''_{ki} = - \int_{\partial\Omega} \int_{\partial\Omega} \frac{1}{R^3} U_k(\mathbf{r}) \cdot [\mathbf{R} \times (\mathbf{n}' \times V_i(\mathbf{r}'))] dS dS', \quad (12)$$

$$c''_k = 4\pi \int_{\partial\Omega} \mathbf{A}_0(\mathbf{r}) \cdot \mathbf{U}_k(\mathbf{r}) dS . \quad (13)$$

The vector functions \mathbf{U}_k and \mathbf{V}_i are orthogonal, $\int_{\partial\Omega} \mathbf{U}_k(\mathbf{r}) \cdot \mathbf{V}_i(\mathbf{r}) dS = 0, \forall k, i$.

At least one integral of the above double integrals can be evaluated analytically. In terms of the unknown \mathbf{A} , (2) is a Fredholm integral equation of the second kind, leading to a well-conditioned matrix of the equations in (5) and (10).

3. TREATMENT OF MULTIPLY CONNECTED REGIONS

In order to present in the simplest way the case of the multiply connected regions, we consider a domain with a shape of a toroid. The toroid hole can be crossed by the electric currents and magnetic fluxes. The circulations of $\nabla \times \mathbf{A}$ and \mathbf{A} along the closed paths Γ' and Γ'' around the toroid and surrounding its hole (as shown in Fig. 2) can be expressed as

$$\int_{\Gamma'} (\nabla \times \mathbf{A}) \cdot d\mathbf{l} = \mu_0 I', \quad \int_{\Gamma''} (\nabla \times \mathbf{A}) \cdot d\mathbf{l} = \mu_0 I'', \quad (14)$$

$$\int_{\Gamma'} \mathbf{A} \cdot d\mathbf{l} = \phi', \quad \int_{\Gamma''} \mathbf{A} \cdot d\mathbf{l} = \phi'', \quad (15)$$

where I' , I'' and ϕ' , ϕ'' are the electric currents and magnetic fluxes through the toroid and through his hole, respectively.

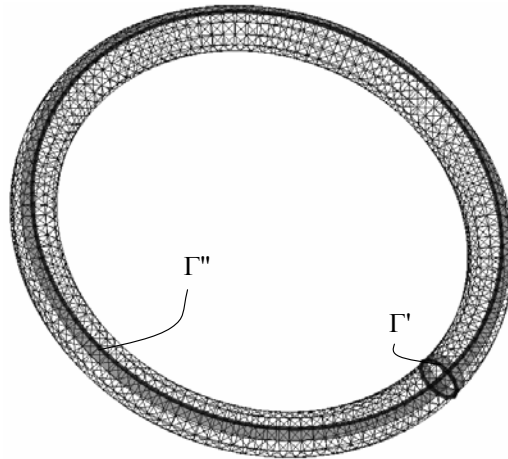


Fig. 2 – Closed paths Γ' and Γ'' around the toroid and surrounding its hole on a discretization mesh.

We define the vector function

$$\mathbf{G}' = \sum_{i \in \{K'\}} \mathbf{V}_i^{(p)}, \quad (16)$$

where $\{K'\}$ is the set of nodes on Γ' and p indicates all the triangular boundary elements on the same side of Γ' that contains at least one node of $\{K'\}$. Similarly, we define the function \mathbf{G}'' associated closed path Γ'' . The following relations replace the form of $\nabla \times \mathbf{A}$ and \mathbf{A} in (3) for the case of multiply connected regions

$$\nabla \times \mathbf{A} = \sum_{i=1}^{N'} \alpha_i \mathbf{V}_i + \mu_0 (\mathbf{G}' I' + \mathbf{G}'' I''), \quad \mathbf{A} = \sum_{i=1}^{N'} \beta_i \mathbf{V}_i + \mathbf{G}' \phi' + \mathbf{G}'' \phi''. \quad (17)$$

For a correct formulation of the vector potential problem on the toroid surface it is necessary to know either the electric currents I' and I'' , or the magnetic fluxes ϕ' and ϕ'' , or a current and a flux. Equations (5) and (10) are completed by projecting equation (2) on the functions \mathbf{G}' and \mathbf{G}'' .

A given electric current through the toroid hole means that the value of I'' is known. If another toroid crosses the hole of the first one, then the current I_1'' that passes through its hole is the same with that passing along the second toroid $I_2' = I_1''$.

The magnetic flux ϕ' along a perfect conductor toroid is equal to zero, as well as I'' . If we impose the value of magnetic flux ϕ'' through his hole, the electric current I' along the toroid results. Then, the self inductance of the toroid is simply obtained from $L = \phi'' / I'$.

4. ILLUSTRATIVE EXAMPLES

In order to prove the accuracy of the proposed method, we firstly choose the perfect conductor domain of a sphere with 1 m radius, placed in a uniform magnetic field $B_0 = 2.51 \times 10^{-2}$ T, oriented along the Oz axis. The sphere surface is approximated by a polyhedral surface with triangular 2,808 facets and 1,406 nodes. In Fig. 3 the magnetic flux density vectors are shown on each triangular element of the sphere boundary. In Fig. 4 the projection of flux density on the (x, y, z) system of coordinates is presented for a meridian located at 30 degrees, depending on latitude. For the perfect conductor sphere an analytical solution can be obtained [9], and, for this example, the weighted mean square error of the computed results is 0.954%.

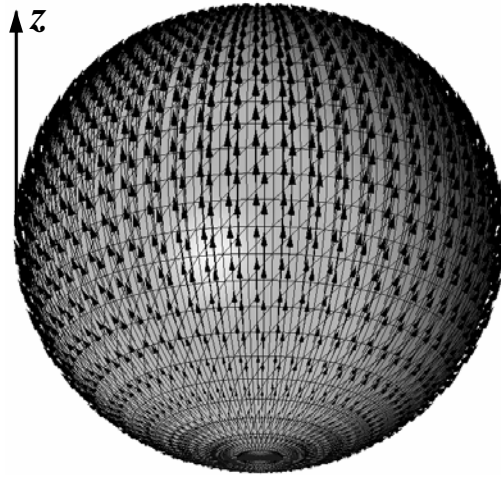


Fig. 3 – \mathbf{B} over the surface of a perfect conductor sphere in a uniform field.

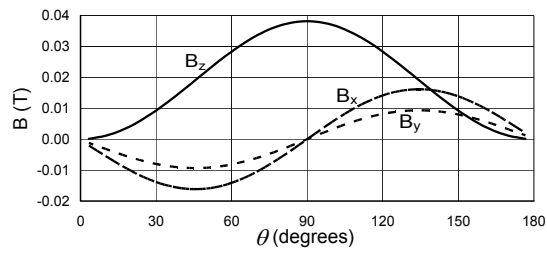


Fig. 4 – Magnetic flux density components *versus* θ at $\varphi = 30^\circ$ for the sphere in Fig. 3.

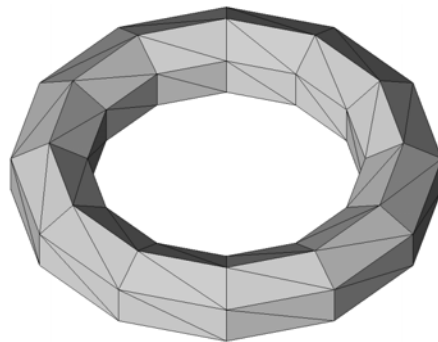


Fig. 5 – A coarse discretization mesh of the toroid surface.

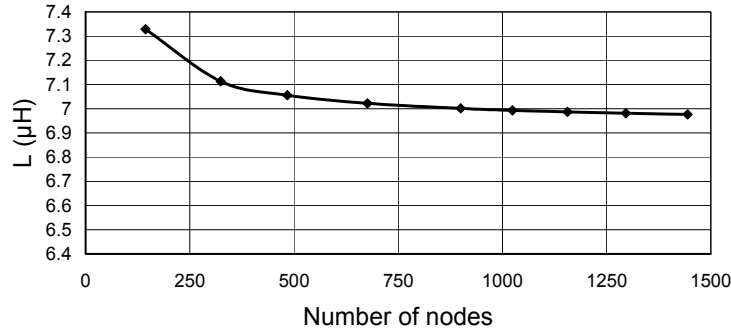


Fig. 6 – Dependence of the self inductance of the toroid in Fig. 5 on the density of the discretization grid.

A second example presents the computation of self-inductance of a perfect conductor toroid with average radius of 4 m and 1 m radius section (see Fig. 5). We choose a magnetic flux equal to unity through the toroid hole ($\phi'' = 1 \text{ Wb}$) and compute I' along the toroid. The toroid surface is discretized in the manner shown in Fig. 5. The computed self inductance is presented for different densities of the discretization mesh (number of nodes) in Fig. 6.

5. CONCLUSIONS

The replacement of the vector potential tangential component boundary condition by imposing only the circulation of \mathcal{A} on any closed paths on $\partial\Omega$ corresponds to the normal component of flux density natural boundary condition on the surface of perfect conductor domains. This "weakening" of restrictions on \mathcal{A}_t allows us to choose the normal component of vector potential to be zero on the boundary and, therefore, results a simplified form of the integral equation (1) and removes the singularity contained in the third right hand side term. If we use the vector shape functions V_i associated with nodes and the test functions V_i and U_i , orthogonal to V_i , we obtain a system of algebraic equations with the number of unknowns equal to twice the number of nodes. In the case of multiply connected domains, we add specialized vector functions associated to the closed paths around the toroids and boarding their holes. The number of such functions is equal to twice the order of connection. In the case of perfect conductor domains, the number of these functions is equal to the order of connectivity.

The method described is particularly effective for determining the inductances of perfect conductor coils. We note that Neumann formula can be used

only to determine the approximate mutual inductance and can not be applied to obtain the self inductance.

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